SYMMETRY IN PHYSICS

VOLUME 1:

PRINCIPLES AND SIMPLE APPLICATIONS

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Preface to Volume 1

One cannot study any physical system for very long before finding regularities or symmetries which demand explanation and, even though the system may be complex, one expects that the regularities will have a simple explanation. This basic optimism, which pervades not only physics but science in general, is justified in the case of symmetries because there is a theory of symmetry which has application in almost all branches of physics and especially in quantum physics. The object of our book is to describe the theory of symmetry and to study its applications in a wide variety of physical systems.

The book has grown out of several lecture courses which we have given at the University of Sussex during the past ten years. One was a general introductory course on symmetry given to third-year undergraduates, one a postgraduate course on symmetry in solid-state physics and one a post-graduate course on symmetry in atomic, nuclear and elementary-particle physics. As a result, the book may be used by students in any of these categories. We regard chapters 1–5 (inclusive) as a minimum selection for any student wishing to study symmetry, although those students who have taken an undergraduate course on linear algebra will find that much of chapter 3 is familiar and may be read quite rapidly. The remaining chapters 6–1[illegible] in volume 1 cover a wide range of applications which is quite sufficient for an undergraduate course. One could even be selective within the first volume by omitting chapters 10–12 on nuclear and elementary particle physics or alternatively by omitting chapters 6 and 9 on the point groups. We would expect the second volume to be used for serious study at the postgraduate level and for occasional reference by the more inquisitive undergraduate.

The first chapter of volume 1 introduces the concept of symmetry with some very simple examples and lists the general consequences. We then leave physics aside for three chapters while preparing the mathematical tools to be used later. The most important of these are group theory and linear algebra which are described in chapters 2 and 3. The fourth chapter brings together these two ideas in a study of group representations and it is this aspect of group theory which is most used in the theory of symmetry. We return to physics in chapter 5 with a brief summary of the basic ideas of quantum mechanics and a general description of the effects of symmetry in quantum systems. The remainder of the book is concerned with applications to different physical systems and the study in greater detail of the relevant groups. We cover a broad range of applications from molecular vibrations to elementary particles and in each case we aim to introduce sufficient background description to enable the reader who has no prior knowledge of that particular physical system to appreciate the role being played by symmetry. Each application is reasonably self-contained and the more sophisticated systems are left until the later chapters. The vibration of molecules is the first phenomenon studied in detail, in chapter 6, and here we are able to illustrate the results of symmetry in classical mechanics before going over to the quantised theory. Chapters 7 and 8 describe the symmetry with respect to rotations with applications to the structure of atoms. It is here that we meet for the first time a continuous group, with an infinite number of elements, or symmetry operations, and the general properties of such groups are described. Chapter 9 describes in some detail the 'point groups', which contain only a finite number of rotations, and uses them to study the influence of a crystal field on atomic states. In chapters 10, 11 and 12 we move on to the more abstract symmetries encountered in nuclear and elementary particle physics but make use of the same general theory that was used for the more concrete applications in earlier chapters. We introduce the groups of unitary transformations in two, three, four and six dimensions and use them to describe the observed symmetry between neutrons and protons and the regularities amongst some of the recently discovered short-lived elementary particles. The ideas of 'strangeness' and 'quarks' are explained.

Volume 2 begins with a further application of the use of 'point groups'—to the motion of electrons in a molecule—and then, in chapter 14, moves away from symmetries with a fixed point to study discrete translations and their applications to crystal structure. The theory of relativity is of profound importance in the philosophy of physics and, when speeds become comparable with that of light, it has practical importance. For all the systems discussed in volume 1 we are able to ignore relativity because the speeds of the particles involved are sufficiently small. Chapter 15 describes the symmetry in four-dimensional space–time which is the origin of relativity theory and discusses its consequences, especially in relation to the classification of elementary particles. The concepts of momentum, energy, mass and spin are interpreted in terms of symmetry using the Lorentz and Poincaré groups and a natural place is found in the theory for particles, like the photon, with zero mass. Chapter 16 is concerned with fields, in contrast to the earlier chapters which dealt with particles or systems of particles. We first describe classical fields, such as the electromagnetic field, using four-dimensional space–time. This is followed by a brief account of the theory of relativistic quantum fields which provides a framework for the creation and annihilation of particles and the existence of antiparticles. Chapters 17 and 18 contain details of two general groups, the 'symmetric' group of all permutations of n objects and the 'unitary' group in N dimensions, and an intimate relation between these two groups is discussed. Particular cases of these two groups have been met earlier. Chapter 19 describes some unexpected symmetries in two familiar potentials, the Coulomb and the harmonic oscillator potentials, and a number of small, unconnected, but interesting topics are collected into the last chapter.

The text includes worked examples and a selection of problems with solutions. A bibliography of references for further reading is given at the end of each chapter for those who wish either to follow the physical applications into more detail or to study some of the mathematical questions to a greater depth.

To aid the reader we have followed the standard convention of using italic type for algebraic symbols such as x, y and z, whereas operators are distinguished by the use of roman type. An operator or matrix will be written T but its matrix elements T_{ij}, which are numbers, will be in italic type. In addition, bold face type will be used for vectors and in chapters 15 and 16 of volume 2 we meet four-vectors $\hat{e}$ which are printed with a circumflex.

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1

Introduction

1.1 The place of symmetry in physics

According to the *Concise Oxford Dictionary*, symmetry is defined as '(Beauty resulting from) right proportion between the parts of the body or any whole, balance, congruity, harmony, keeping'. Although there is much complex detail in physics there is also much beauty and simplicity and it is the symmetry in physical laws and physical systems which is largely responsible for this. Consequently, symmetry plays an important role in physics and one which is increasing in importance with modern developments. It is the purpose of this book to explain in general terms why the existence of symmetry leads to a variety of physical simplicities in both classical and quantum mechanics. To illustrate the general results we shall refer to simple properties of molecules, crystals, atoms, nuclei and elementary particles. Although these physical systems are so obviously different from one another, nevertheless the same theory of symmetry may be applied to them all. The study of symmetry, therefore, helps to unify physics by emphasising the similarity between different fields.

It is true that symmetry plays a part in both classical and quantum physics, but it is in the latter that most interest lies. There are several reasons for this. The first is that there is a much greater scope for symmetry to exist in the microscopic domain since, for example, one electron is identical with any other electron and one atom of carbon (say) is identical with any other. The second reason is that at the microscopic level one must use quantum mechanics which is inherently more complicated than classical mechanics and so provides more scope for simplification through symmetry arguments. For example, a particle is described by a wave *function* rather than a single position. One further reason is that the structure of atomic and subatomic systems is now one of the exciting frontiers of science and the ideas of symmetry are helping to create order out of apparent chaos.

Throughout physics one uses mathematics as the tool with which to investigate the consequences of some assumed theory or model. For example, in the motion of a particle of mass M in one dimension x under some force $f(x)$ the physical law (Newtonian theory) tells us that $f(x) = M(\mathrm{d}^2x/\mathrm{d}t^2)$. To find the position $x(t)$, as a function of time, given $f(x)$, we must solve this differential equation, putting in the initial values of x and $\mathrm{d}x/\mathrm{d}t$. Thus, in Newtonian mechanics, the differential and integral calculus is the appropriate tool. In studying the symmetry of physical systems we are asking about their behaviour under transformations. For example, if a particle moves in one dimension under the influence of a potential $V(x)$, that potential may have reflection symmetry in the origin, i.e. $V(-x) = V(x)$. In this case the potential is said to be invariant (unchanged) under the transformation which replaces x by $-x$. In another example, that of a particle moving in three dimensions, the potential may have spherical symmetry, which means that, in spherical polar coordinates, the potential is independent of angle and may be written $V(r)$. In this case the potential is invariant under any of the transformations which rotate through any angle about any axis through the origin—an infinite number of transformations!

To investigate the physical consequences of the symmetry of a system we must, therefore, learn something about transformations and in particular about the set (collection) of transformations which leave some function, like the potential, invariant. The theory of such sets of transformations is called 'group theory' by mathematicians and this is the appropriate tool for the physicist to use in studying symmetry.

It is fascinating to draw an analogy between the use of calculus in classical mechanics and the use of group theory in quantum mechanics. Historically the discovery of Newton's laws and the invention of the calculus occurred at about the same time in the seventeenth century. Although the ideas of group theory were introduced into mathematics as early as 1810 it was not until the 1920s that the theory of group representations, which is crucial to the study of symmetry, was developed. This was the very time when physicists were formulating the quantum theory. In fact the significance of symmetry in quantum mechanics was realised very early in the classic works of E. Wigner, in 1931, H. Weyl, in 1928, and Van-der-Waerden, in 1932.

There have always been those who have argued that it is unnecessary to use group theory in quantum mechanics. In a sense this is true, since group theory itself is built from elementary algebraic steps. However, the investment of effort in learning to use the sophisticated tool which is group theory is soon rewarded by handsome dividends of simplification and unification in the study of complex quantum mechanical systems. After all, one could argue that the calculus is not necessary in classical mechanics. For example geometrical arguments could be used to show that the inverse square law of gravitational attraction leads to elliptical orbits. In fact, Newton originally used such a method but in modern times we understand this result through the solution of a differential equation. Looking ahead, it is exciting to speculate that further major advances in mathematics and physics may go hand in hand in the future.

1.2 Examples of the consequences of symmetry

To whet the appetite we now list a number of physical systems which possess symmetry and we point out some features of their behaviour which are direct consequences of the symmetry. Simpler examples are given first. Although in some cases we are able to relate the behaviour to the symmetry without developing new methods this is, of course, not always possible. It is the purpose of this book to describe generally the consequences of symmetry and it will not be until much later in the book that we shall be in a position to understand and to predict the behaviour of systems with intricate symmetries.

1.2.1 One particle in one dimension (classical)

A particle of mass M, moving in one dimension under the influence of a potential $V(x)$, will have its motion governed by the equation

$$M\ddot{x} = -\mathrm{d}V/\mathrm{d}x \tag{1.1}$$

Suppose now that $V(x)$ is a constant, independent of x; in other words that it is invariant under translation. Then clearly $M\ddot{x} = 0$ and, integrating, gives $M\dot{x} = C$, showing the conservation (constancy) of linear momentum $M\dot{x}$.

1.2.2 One particle in two dimensions (classical)

In two dimensions the motion of the particle is governed by the two equations

$$M\ddot{x} = -\partial V/\partial x \quad \text{and} \quad M\ddot{y} = -\partial V/\partial y \tag{1.2}$$

Suppose now that $V(x, y)$ is invariant with respect to rotation about the origin; in other words that $V(x, y)$ is independent of the polar angle θ if expressed in terms of the polar coordinates r, θ rather than the cartesian x and y. In this case $\partial V/\partial\theta = 0$. However,

$$\frac{\partial V}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial V}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial V}{\partial y} = -y\frac{\partial V}{\partial x} + x\frac{\partial V}{\partial y}$$

and using equation (1.2)

$$\frac{\partial V}{\partial \theta} = M(y\ddot{x} - x\ddot{y}) = M\frac{\mathrm{d}}{\mathrm{d}t}(y\dot{x} - x\dot{y})$$

so that the invariance $\partial V/\partial\theta = 0$ implies the constancy of the quantity $M(y\dot{x} - x\dot{y})$ which is the moment of momentum (or angular momentum) about an axis through the origin and perpendicular to the plane.

If the particle were free to move in three dimensions in a potential which was invariant with respect to rotations about *any* axis then this argument shows that any component of the angular momentum is constant. In other words, for a spherically symmetric potential, both the magnitude and the direction of the angular momentum are conserved.

1.2.3 Two particles connected by springs (classical)

Two particles of equal mass M are connected to each other and to fixed supports by equal collinear springs with spring constant λ. Let the natural length of the springs be a and the supports a distance $3a$ apart. Denote the displacements of the two particles from their equilibrium positions by x_1 and x_2. Although the general displacement, illustrated in figure 1.1, has no

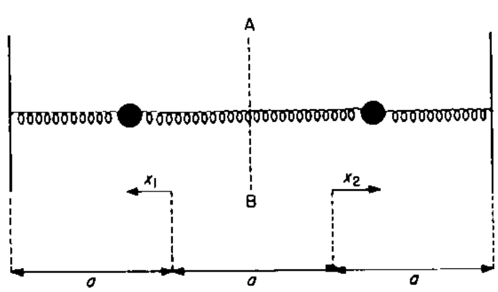

Figure 1.1

symmetry it is intuitively clear that, in some sense, the system has reflection symmetry about the centre. In fact, both the kinetic and potential energies

$$T = \tfrac{1}{2}M(\dot{x}_1^2 + \dot{x}_2^2) \quad \text{and} \quad V = \tfrac{1}{2}\lambda\{x_1^2 + x_2^2 + (x_1 + x_2)^2\}$$

are invariant with respect to the interchange of x_1 and x_2, which is the transformation of coordinates x_1 and x_2 produced by a reflection in the line AB.

The consequences of symmetry are not very dramatic in this case, but the generalisation to the vibration of atoms about their equilibrium positions in a molecule is of considerable importance. It is therefore worth while to solve