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The unirationality of the moduli space of étale double covers of genus five curves

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To my family, with love.

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E qui chiudo.

Contents

Introduction

有志者事竟成
Nothing is impossible
to a willing mind.

Moduli Spaces and Unirationality

Let X be an algebraic variety over a field $\mathbb{K}$. We say that X is *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$ from a projective space to X. As Kollár, Smith and Corti [29] point out, here the use of the prefix "uni" refers to the map from $\mathbb{P}^n$ to the variety, defined in one direction only in contrast with the two mutually inverse rational maps required for rationality.

Already in 1915 Severi proved that the moduli space $\mathcal{M}_g$ of curves of genus g is unirational for $g \leq 10$—see Arbarello and Sernesi, [2], for a modern rigorous proof—and conjectured the same to hold for every genus; but this was to remain open until the beginning of the 1980s, when Harris and Mumford [27] disproved it by showing that over the complex numbers, $\mathcal{M}_g$ is of general type if g is odd and $g \geq 25$. Meanwhile Sernesi [48] proved the unirationality of $\mathcal{M}_{12}$, thus leaving a gap at genus 11 that was filled later by Chang and Ran [6], who also proved that $\mathcal{M}_{13}$ is unirational. The question of what happens for large g was settled when Eisenbud and Harris [17] published their proof that $\mathcal{M}_g$ is of general type for all $g \geq 24$ and $\mathcal{M}_{23}$ has Kodaira dimension at least 1.

A rather similar story unfolded for $\mathcal{A}_g$, the moduli space of principally polarised Abelian varieties of dimension g. Tai [52] proved that $\mathcal{A}_g$ is of general type for $g \geq 9$: this was improved to $g \geq 8$ by Freitag [19] and to $g \geq 7$ by Mumford [41]. This was the state of knowledge when Donagi [12] proved that

$\mathcal{A}_5$ is unirational. His proof is related to work by Clemens [8], who proved that $\mathcal{A}_4$ is unirational, using intermediate Jacobians. Very recently Grushevsky and Lehavi [25] have shown that $\mathcal{A}_6$ is of general type.

However, Donagi proved something else: he also showed the unirationality of $\mathcal{R}_6$, the moduli space of étale double covers of curves of genus six. Other proofs of this were given by Verra [54] and by Mori and Mukai [38]. It follows from this that $\mathcal{A}_5$ is unirational, because of the generic surjectivity (also shown by Donagi) of the *Prym map*.

Let $\pi\colon \widetilde{C} \to C$ be an étale double cover of a smooth complete algebraic curve C of genus $g \geq 2$, and let $\operatorname{Jac}(\widetilde{C})$ and $\operatorname{Jac}(C)$ be the Jacobians. As Mumford [40] explains, the Jacobian $\operatorname{Jac}(\widetilde{C})$ splits under the action of the involution $\iota\colon \widetilde{C} \to \widetilde{C}$ interchanging the two sheets of the cover, into a positive part and a negative part P, the Prym variety. The principal polarisation of $\operatorname{Jac}(\widetilde{C})$ induces twice a principal polarisation on P and it is a fact that the dimension of P is $g-1$. By associating to a double cover π its Prym variety we define the *Prym map*, that is a morphism

$$p_g\colon \mathcal{R}_g \longrightarrow \mathcal{A}_{g-1}$$

where $\mathcal{R}_g$ denotes the moduli space of étale double covers of curves of genus g, or equivalently the moduli space of curves $C \in \mathcal{M}_g$ equipped with a (nonzero) line bundle $\mathscr{L}$ whose square is trivial. Thus $\mathcal{R}_g$ is also equipped with a morphism to $\mathcal{M}_g$, which is a finite cover of degree $2^{2g}-1$.

It is known [see Beauville, 4] that for $g \geq 7$ the Prym map is generically injective, while for $g \leq 6$ it is generically surjective. Thus in proving that $\mathcal{R}_6$ is unirational, Donagi [12] was able to show the unirationality of $\mathcal{A}_5$ using the Prym map. At about the same time Catanese [5] proved that $\mathcal{R}_4$ is rational. Since then there has been a gap between four and six, which this thesis proposes to fill; in fact the same result will provide another proof of the unirationality of $\mathcal{A}_4$, substantially different from the one in Clemens [8].

The Prym map has been described extensively by Donagi and Smith [14] both in genus 6 and in genus 5. Classically p_6 was known to be generically finite, and quite famously they proved that its degree is 27. Later Donagi [13] also showed that the general fibre of p_6 carries the structure of the configuration of lines on a cubic surface, and gave a complete description of the two-dimensional fibres of p_5.

The Genus Five Case

Let X be a hypersurface in $\mathbb{P}^3$ of degree four, whose only singularities are six isolated ordinary double points $P_0, \ldots, P_5$. Working in coordinates and realising $\mathbb{P}^3$ as the projective spectrum of the polynomial ring $R = \mathbb{K}[x_0, \ldots, x_3]$, assume the point P_0 to be $[0:0:0:1]$ and X to be given by $\operatorname{Proj} R/(F)$ where F is a form of degree four contained in $(x_0, x_1, x_2)^2$. In particular F can be written as follows

$$F = u_2 x_3^2 + 2u_3 x_3 + u_4$$

where u_d is a form of degree d in $\mathbb{K}[x_0, x_1, x_2]$. To such a surface we can associate a plane curve C_X as follows. Realise $\mathbb{P}^2$ as the set of all the lines through P_0, more precisely fix a hyperplane in $\mathbb{P}^3$ not containing P_0 and projecting define a morphism $\mathbb{P}^3 \setminus \{P_0\} \to \mathbb{P}^2$ whose fibres (over the closed points) are all the lines through the point P_0. Composing with the closed immersion of $X \setminus \{P_0\}$ in the punctured projective space defines a morphism $\pi \colon X \setminus \{P_0\} \to \mathbb{P}^2$ with finite fibres (in fact, generically finite). The curve C_X will be the scheme of lines through P_0 that are tangent to X away from P_0, which is a closed subscheme of the locus of points in $\mathbb{P}^2$ whose fibre under π consists of one point only. More precisely, working in coordinates as above, it is given by the following projective spectrum

$$C_X = \operatorname{Proj} \mathbb{K}[x_0, x_1, x_2]/(u_3^2 - u_2 u_4)$$

Furthermore the generic surface X gives rise to a curve C_X whose only singularities come from singularities of X, in other words generically the curve C_X has precisely five double points $P_1, \ldots, P_5$, and therefore its arithmetic genus is five.

Following Clemens [8] we can now associate to the surface X a (quartic) *double solid*, that is a threefold Λ_X endowed with a two-sheeted covering $p \colon \Lambda_X \to \mathbb{P}^3$ branched along the surface X. This can be realised as the closed subscheme of the *weighted projective space* $\mathbb{P}(1,1,1,1,2)$ given by the equation $F - t^2$, in other words

$$\Lambda_X = \operatorname{Proj} \mathbb{K}[x_0, \ldots, x_3, t]/(F - t^2)$$

where the degree of x_i is considered to be 1, and the degree of t to be 2. According to Kreussler [31] Λ_X is birational to a conic bundle over $\mathbb{P}^2$: more precisely the unique morphism of schemes f which makes the following diagram commute

$$\begin{array}{ccccc} \mathrm{Bl}_{P_0}(\Lambda_X) & \longrightarrow & \Lambda_X & \xrightarrow{p} & \mathbb{P}^3 \\ & & & \searrow^{f} & \downarrow{\scriptstyle \pi} \\ & & & & \mathbb{P}^2 \end{array}$$

is a conic bundle over $\mathbb{P}^2$ for which the curve C_X is the locus of points whose fibre is degenerate. Blowing up in the remaining five points we construct a conic bundle over a degree 4 del Pezzo surface containing the canonical model of the curve C_X, which is given by the locus of points whose fibre consists of two lines meeting transversally, and comes therefore naturally endowed with a double cover.

Let $\mathcal{Q}$ be the space of quartic hypersurfaces in $\mathbb{P}^3$ with six isolated ordinary double points, one of which is marked. To see that this is unirational fix five points $P_1, \ldots, P_5$ in $\mathbb{P}^3$ and consider the space B_0 of all quartic hypersurfaces with double points at $P_1, \ldots, P_5$ and another at P_0, where P_0 is not fixed. Each fibre of the morphism from B_0 to $\mathbb{P}^3$ associating to each surface the sixth node P_0 is just a linear space, therefore B_0 is a projective vector bundle over an open subset of $\mathbb{P}^3$ and hence is rational. This proves that the construction above defines a morphism from a unirational variety to $\mathcal{R}_5$, which is in turn endowed with a finite (in fact 1023-to-1) natural projection to $\mathcal{M}_5$, the moduli space of curves of genus five. Since $\mathcal{R}_5$ is known to be irreducible, to prove the unirationality it is now enough to prove that the map to $\mathcal{M}_5$ is generically surjective

We now have a family of canonically embedded curves of genus five, each one given by the intersection of three quadrics in $\mathbb{P}^4$. Following the notes on deformations of schemes by Sernesi [49], we fix a central fibre in this family and with *Macaulay 2* [21] we compute explicit equations for it, whose coefficients depend on 13 parameters. The computation of the Kodaira-Spencer map is now reduced to linear algebra, in particular we exploit the property of our curves of being canonically embedded, meaning that two of them will be isomorphic if and only if they are projectively equivalent. In other words we are now looking at the Grassmannian $\mathrm{Gr}(3, 15)$ of 3-dimensional linear spaces of quadrics in $\mathbb{P}^4$, more precisely at its tangent space at the point described by our central fibre. Taking into account the action of $\mathrm{SL}(5)$ describing classes of isomorphism, our computation reduces to computing the rank of a big matrix and our result is that this rank is maximal.

It is likely that the need for direct computation could be reduced, but we took this approach for practical reasons. Note that Donagi [12] also relies upon computer calculations.

The Structure of this Thesis

This thesis is divided into four chapters. Notation is fixed in the first one by reviewing some standard basic topics in algebraic geometry, in particular schemes. After a brief description of Abelian varieties, with some emphasis on Jacobians and Pryms, the chapter ends with an original argument for the unirationality of $\mathcal{A}_4$, the moduli space of principally polarised Abelian varieties of dimension four.

The second chapter is devoted to the description of the geometric configuration concerning quartic hypersurfaces X in $\mathbb{P}^3$ with at least one ordinary double point P. After the definition of the associated plane sextic C_X we analyse its singularities, showing that a surface X which is generic in a very precise sense gives rise to a curve C_X whose singularities are in one-to-one correspondence with the singularities of X other than P. Quartic double solids are introduced, and a comparative discussion about equivalent possible definitions is provided, before we state Kreussler's theorem [31] about double solids and conic bundles. But the main result is the following.

Proposition (2.8). *Let X be a generic quartic hypersurface in $\mathbb{P}^3$ with six isolated ordinary double points $P_0, \ldots, P_5$, and let $\widetilde{C_X}$ be the canonical model of the associated plane sextic with respect to P_0. Then there exists a natural unbranched double cover of the curve $\widetilde{C_X}$.*

The possibility that the double cover might be trivial, that is consisting of two copies of the curve $\widetilde{C_X}$, is not ruled out directly, but it is shown to be equivalent to the reducibility of the locus of degeneracy of a conic bundle W_X. This leads to a description of an open condition over the moduli space of quartic surfaces in $\mathbb{P}^3$, which a direct computation shows to be nonempty.

Finally we describe a general method to compute an explicit example, providing some background to better explain the results obtained with *Macaulay 2* later in the thesis. In particular we write down a homogeneous polynomial of degree four providing the equation for our standard test surface $\mathbb{X}$.

Chapter 3 is all about moduli spaces. We review the definition of fine and coarse moduli space, briefly discussing the Hilbert scheme as an example of a fine moduli space, and describe how to define a morphism using natural transformations. We then show that the variety $\mathcal{Q}$, parametrising quartic hypersurfaces in $\mathbb{P}^3$ with six isolated ordinary double points, one of which is marked, up to projective equivalence, is unirational and 13-dimensional; and define a morphism $\mathcal{Q} \to \mathcal{R}_5$. With the main result in mind we then look for sufficient conditions for this morphism to be generically surjective.

Theorem (3.14). *The moduli space $\mathcal{R}_5$ of étale two-sheeted coverings of curves of genus five is unirational.*

The computation of the differential of the morphism $\mathcal{Q} \to \mathcal{M}_5$, obtained from the one above by composition with $\mathcal{R}_5 \to \mathcal{M}_5$, at one given point $\mathbb{X}$ is reduced to the computation of the rank of a 46×45 matrix, and the theorem follows if it is shown that this rank is maximal for a generic quartic surface. Again we claim that the test curve defined in Chapter 2 satisfies this condition.

To complete the proof of the theorem, and to show that all the open sets we have considered so far are non-empty, we provide in Chapter 4 the results obtained with *Macaulay 2* when performing the computations described earlier in the thesis.

Chapter 1

Basic Material

After a very brief introduction to schemes we describe Abelian varieties, with some emphasis on Jacobians and Pryms. The former is done to fix notation while the latter to provide some context for our main result. In fact the chapter will end with an original argument for the unirationality of $\mathcal{A}_4$, the moduli space of principally polarised Abelian varieties of dimension four.

1.1 Schemes

The concept of a *scheme* naturally arises from the observation that an affine variety and its coordinate ring are essentially the same object. In fact, more than being just a remark, this awareness is an intrinsic part of the theory from the very beginning, whether expressed in classical style, as for instance on Shafarevich [50] or in a more modern way as on the two undergraduate texts by Miles Reid [43, 44]. In great generality the *prime spectrum* of a ring A is defined to be the set $\operatorname{Spec} A$ of all its prime ideals: thus for example consider the spectrum of the ring of polynomials over a field $\mathbb{K}$, that is

$$\mathbb{A}^n_{\mathbb{K}} := \operatorname{Spec} \mathbb{K}[X_1, \ldots, X_n].$$

Famously Hilbert's Nullstellensatz states that $\mathbb{K}$ is algebraically closed if and only if the set of maximal ideals in this spectrum is in bijection with n-tuples in $\mathbb{K}^n$, so it makes sense to call this abstract set *affine space*. The Zariski topology is introduced in the classic way, only in this context it is more natural; indeed closed subsets—denoted $\mathcal{V}(\mathfrak{a})$—will consist of prime ideals lying *above* a given ideal $\mathfrak{a} \subseteq A$, and a base for the topology will be given by open sets cut out by principal ideals, in other words by sets $D(\alpha)$ defined to be the complement of $\mathcal{V}(\alpha)$, for some element $\alpha \in A$. Finally the main ingredient is provided by sheaf theory, as it always is according to Tennison [53], who points out that several types of manifolds can be obtained by specialising the abstract concept of *geometric space*. In the case of schemes this is achieved by defining $\mathscr{O}(D(\alpha))$ to be the localised ring A_α for any distinguished open subset $D(\alpha)$. A *scheme* is then a geometric space locally isomorphic to the spectrum of a ring, in particular it is a pair $(X, \mathscr{O}_X)$ consisting of a topological space X and a sheaf of rings over X, satisfying the following properties:

- there exists an open *affine* covering of X, consisting of open subsets U such that the pair $(U, \mathscr{O}_X|_U)$ is isomorphic to Spec $\mathscr{O}_X(U)$;
- for any point $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring;
- for any open subset $U \subseteq X$, the pair $(U, \mathscr{O}_X|_U)$ is again a scheme.

The natural identification between an affine variety and its coordinate ring becomes in this way a precise statement: the category of affine schemes is equivalent to the category of commutative rings with identity, with arrows reversed. This equivalence is given by the contravariant functor Spec: more precisely Eisenbud and Harris [18, Theorem I-40] prove it is a consequence of the adjunction property of Spec.

There are essentially two possible ways to define projective space $\mathbb{P}^n_{\mathbb{K}}$: one consists in gluing together $n+1$ copies of affine n-space and the other is to introduce an equivalence relation on $\mathbb{A}^{n+1}_{\mathbb{K}}$. Both these ideas are encoded in the theory of schemes: the former is a fundamental procedure sometimes called *gluing lemma* and described in detail by Shafarevich [51, §V.3.2], and the latter amounts to a general construction analogous to Spec, the *homogeneous spectrum of a graded ring*. Given a graded ring $S = \bigoplus_{d \geq 0} S_d$, Liu [33, §2.3.3] defines Proj S to be the set of all homogeneous prime ideals $\mathfrak{p}$ which do not contain all of $S_+ = \bigoplus_{d \geq 1} S_d$, and introduces in this set the topology induced by the inclusion Proj $S \subset$ Spec S. Thus a closed subset of Proj S will be given by the intersection $\mathcal{V}_h(E) = \mathcal{V}(E) \cap \text{Proj}\, S$, where E could be in principle any subset of

S but in fact can be assumed to contain only homogeneous elements of positive degree. Again open sets $D_h(\alpha)$ corresponding to sets E with just one element form a base for the Zariski topology. Observe also that $D_h(\alpha)$ coincides with the intersection $D(\alpha) \cap \operatorname{Proj} S$. Then, if $S_{(\alpha)}$ denotes the subring of elements of degree zero in the localised ring S_α, $\operatorname{Proj} S$ can be endowed with a unique structure of a scheme such that for any homogeneous $\alpha \in S_+$, the open set $D_h(\alpha)$ is affine and isomorphic to $\operatorname{Spec} S_{(\alpha)}$ [Liu, 33, Proposition 2.3.38]. Projective space is thus defined to be the homogeneous spectrum of the polynomial ring, that is

$$\mathbb{P}^n_{\mathbb{K}} := \operatorname{Proj} \mathbb{K}[X_0, \dots, X_n]$$

and each of the $n+1$ copies of affine space that cover it is given by $D_h(X_i)$. In fact it is possible to state a general characterisation for a union of basic open sets to cover $\operatorname{Proj} S$.

Proposition 1.1. *Let E be any subset of S consisting of homogeneous elements of positive degree, and let $\mathfrak{a}$ be the homogeneous ideal generated by E. Then the following are equivalent:*

$$(a)\ \bigcup_{\alpha \in E} D_h(\alpha) = \operatorname{Proj} S; \qquad (b)\ \mathcal{V}_h(\mathfrak{a}) = \varnothing; \qquad (c)\ S_+ \subseteq \sqrt{\mathfrak{a}}.$$

Proof. Recall that the radical $\sqrt{\mathfrak{a}}$ of a homogeneous ideal $\mathfrak{a}$ is homogeneous, and that the complement of the closed subset $\mathcal{V}_h(\mathfrak{a})$ is given by the union of $D_h(\alpha)$ where α runs through a set of generators of $\mathfrak{a}$. □

Projective space can in fact be constructed *over* any ring A by simply considering the projective spectrum of $A[X_0, \dots, X_n]$, and more generally over any scheme by means of a *fibred product*. In this case it is written $\mathbb{P}^n_B$ where B is either the *base scheme* or the base ring. By fibred product we mean here the familiar notion from category theory—see Mac Lane [34, §III.4]—sometimes called also pull-back and described by a well known universal property. As Mumford writes [42, §II.2], "there is one exceedingly important and very elementary existence theorem in the category of (pre)schemes. This asserts that arbitrary fibred products exist." When all the schemes involved are affine, say $B = \operatorname{Spec} A$ is the base and $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} T$ are spectra of A-algebras, the fibred product is also affine and is given by

$$X \times_S Y = \operatorname{Spec}(R \otimes_A T)$$

In this context the word *projective* assumes a relative meaning: a scheme X is projective over B if there exists a *closed immersion* $X \hookrightarrow \mathbb{P}^n_B$. In the equivalence of categories between affine schemes and commutative rings, closed immersions correspond to surjective homomorphisms, and closed subschemes correspond to ideals; but since Proj is not even a functor some work is required to establish the next result, which we quote from Liu [33, Proposition V.1.30].

Proposition 1.2. *Let A be a ring. A scheme X over A is projective if and only if it is isomorphic to* $\operatorname{Proj} S/I$*, where $S = A[X_0, \ldots, X_n]$ and I is a homogeneous ideal of S contained in S_+.*

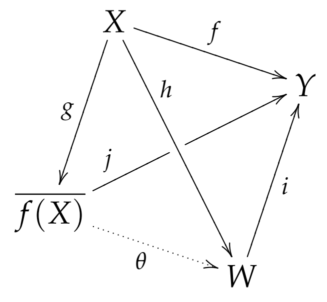
In this context a *projective algebraic variety* will be a projective scheme over a field $\mathbb{K}$, usually assumed to be algebraically closed, defined by a homogeneous prime ideal $\mathfrak{p}$ inside the polynomial ring $\mathbb{K}[X_0, \ldots, X_n]$.

If $f\colon X \to Y$ is any continuous map, we can factor it through the closed subset $\overline{f(X)}$ of Y as a dominant morphism followed by a closed immersion. The same factorisation is possible in the set-up of schemes, although not in complete generality. For instance if X and Y are affine schemes, f is induced by a ring homomorphism $\varphi\colon B \to A$ which will factor through the sub-ring of A given by the image of φ as follows

$$B \longrightarrow \operatorname{Im}\varphi \longrightarrow A$$

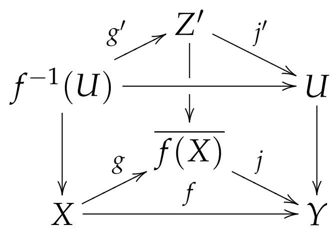
where $B \to \operatorname{Im}\varphi$ is surjective, and $\operatorname{Im}\varphi \to A$ is injective. This purely algebraic configuration corresponds precisely to the geometric one, indeed injective homomorphisms induce dominant maps and surjective homomorphisms induce closed immersions—see Hartshorne [28, Exercise II.2.18]. We call $\operatorname{Spec}\operatorname{Im}\varphi$ the *scheme-theoretic image of f*, and we denote it $\overline{f(X)}$ in analogy with the topological construction. The following result can be found in Hartshorne [28, Exercise II.3.11] or Eisenbud and Harris [18, Definition V-2] or also Liu [33, Exercise II.3.17].

Theorem 1.3. *Let $f\colon X \to Y$ be a quasi-compact morphism of schemes. Then there exists a scheme $\overline{f(X)}$, endowed with a closed immersion $j\colon \overline{f(X)} \to Y$ and a dominant morphism $g\colon X \to \overline{f(X)}$ such that $f = jg$ and with the following universal property*



for every other scheme W endowed with a closed immersion i and a morphism h such that $f = ih$ there exists a unique closed immersion θ such that $i\theta = j$.

Furthermore, for every open subset $U \subseteq Y$ we have the following commutative diagram, where $Z' = \overline{f\big(f^{-1}(U)\big)}$ and vertical arrows are open immersions



That is, the scheme-theoretic image of the restricted morphism $f|^U$ is naturally endowed with an open immersion in the scheme-theoretic image of f.

Proof. If f is a morphism of affine schemes the above argument completes the proof, the universal property being easily checked (note that $\overline{f(X)}$ is necessarily affine). If $Y = \operatorname{Spec} A$ is affine, then f is adjoint to the homomorphism $f^\#_Y\colon A \to \Gamma(X, \mathscr{O}_X)$ and if such a scheme exists it must be affine. Since f is quasi-compact, X can be covered by a finite number of open affine subsets, say $V_i = \operatorname{Spec} B_i$; the composition $V_i \to X \to A$ is induced by the composition of ring homomorphisms $r_i \circ f^\#_Y$ where $r_i\colon \Gamma(X, \mathscr{O}_X) \to B_i$ is the restriction homomorphism of the sheaf $\mathscr{O}_X$, let $\mathfrak{a}_i$ be its kernel and let $\mathfrak{a}$ be the sum of the $\mathfrak{a}_i$. Now the affine scheme $\operatorname{Spec} A/\mathfrak{a}$ comes endowed with a closed immersion and a dominant morphism as required, and the universal property is easily checked (again $\overline{f(X)}$ is necessarily affine).

Let $U = D(\alpha)$ be an open basic subset of Y, which we still assume to be affine, then as in Hartshorne [28, Exercise II.2.16] $f^{-1}(U) = X_\alpha$ and for each i $X_\alpha \cap V_i$ is affine and given by $\operatorname{Spec} B_{i\alpha}$. This proves that, when Y is affine and U is an open basic subset of Y, there exists an open immersion $Z' \to \overline{f(X)}$ as in the statement. If now we let U to be any open affine subset of Y, we can cover U by basic open sets—which will be basic open sets for both Y and U—and construct a collection of open immersions for both $\overline{f(X)}$ and Z', indexed on the open covering of U and therefore covering Z'. Thus we construct the open immersion $Z' \to \overline{f(X)}$ by a gluing argument.

To complete the proof we need another gluing argument. When X and Y are not affine, we take an open affine covering $\{U_i\}$ of Y and construct the family of schemes $\{(Z_i, g_i, c_i)\}$ given by the scheme-theoretic images of the restrictions $f|^{U_i}$, where g_i is a dominant morphism and c_i is a closed immersion. In order to apply the Gluing Lemma on any of these schemes Z_i we define for each j the open subscheme $Z_{ij} \subseteq Z_i$ as $c_i^{-1}(U_i \cap U_j)$. For any open affine subset of $U_i \cap U_j$ we construct the scheme-theoretic image of the restriction of f and obtain an open immersion in both Z_{ij} and Z_{ji}, defining eventually an isomorphism. □

Since any morphism of varieties is of finite type, and any morphism of finite type is quasi-compact, this construction makes perfect sense for any morphism between algebraic varieties. Quite surprisingly however we can prove something more, namely the image of a variety is again a variety.

Corollary 1.4. *Let $f \colon X \to Y$ be a quasi-compact morphism of schemes. When X is reduced the scheme-theoretic image of f is a reduced scheme, while when X is an integral scheme it is integral.*

Proof. Let Z be the scheme-theoretic image of f as in the Theorem, and let $\widetilde{Z}$ be the reduced scheme associated to Z as in Hartshorne [28, Exercise II.2.3] with closed immersion $r \colon \widetilde{Z} \to Z$. Since X is reduced, there exists a unique morphism $h \colon X \to \widetilde{Z}$ such that $rh = g$, and if we let $i = jr$ we obtain $ih = jrh = jg = f$. Therefore by the universal property of Z there exists a unique θ such that $i\theta = j$, which means $jr\theta = j$. Since closed immersions are monic this implies $r\theta = \mathrm{id}_Z$. Viceversa we have $jr\theta r = i\theta r = jr$ therefore $r\theta r = r$ which implies $\theta r = \mathrm{id}_{\widetilde{Z}}$. In conclusion Z is isomorphic to $\widetilde{Z}$, hence it is a reduced scheme.

Assume now X is an integral scheme, then all the restriction homomorphisms of $\mathscr{O}_X$ are injective. Therefore with reference to the proof of the Theorem when $Y = \operatorname{Spec} A$ is affine the scheme theoretic image of f is given by $\operatorname{Spec} A/\mathfrak{a}$ where $\mathfrak{a}$ is the kernel of the morphism $A \to \Gamma(X, \mathscr{O}_X)$ inducing f. The ring $\Gamma(X, \mathscr{O}_X)$ is a domain, therefore the scheme theoretic image is an integral scheme. □

1.2 Abelian Varieties

Roughly speaking an *Abelian variety* is a projective algebraic variety V which is also a group. However to make this statement precise one needs to be careful because of the abstract nature of algebraic varieties. The starting point is the concept of *group variety*, which is an algebraic variety over an algebraically closed field $\mathbb{K}$ whose set of $\mathbb{K}$-rational points is a group; but even this can be made more general: see Milne [36]. The group axioms will be phrased in terms of properties of certain morphisms via category theory, with the identity element being itself a morphism from $\operatorname{Spec} \mathbb{K}$ to V. Abelian varieties are then defined to be complete connected group varieties, and it is proved [ibid., Corollary 1.4 and §5] that they are projective and commutative.

Over the complex numbers an algebraic variety can be regarded as a particular kind of complex manifold (Kähler), and can be studied using the machinery of differential geometry. This leads to some extent to a dichotomy in the theory, perfectly exemplified by the case of Abelian varieties: over $\mathbb{C}$ these are indeed *complex tori*, that is quotients $\mathbb{C}^g/\Gamma$ for some full lattice Γ [Mumford, 39]. As a consequence it is possible to state and prove all the general results about Abelian varieties in terms of rather simple complex functions [theta functions, see Debarre 10] without developing the more abstract theory; however not every complex torus is isomorphic to a projective variety, and one has to look for a *polarization*.

Theorem 1.5 (Riemann conditions)**.** *A complex torus $\mathbb{C}^g/\Gamma$ is isomorphic to a projective variety if and only if there exists a skew-symmetric form $E\colon \Gamma \times \Gamma \to \mathbb{Z}$ such* *that*

(a) $E_{\mathbb{R}}(iv, iw) = E_{\mathbb{R}}(v, w)$;

(b) $E_{\mathbb{R}}(v, iv) > 0$, *for every* $v \neq 0$.

where $E_{\mathbb{R}}$ *is the skew-symmetric* $\mathbb{R}$*-bilinear form over* $\mathbb{C}^g$ *obtained from* E *by extension, since* $\Gamma \otimes \mathbb{R} = \mathbb{C}^g$.

E is usually called a *Riemann form*, to give one is equivalent to give a Kähler differential form on the complex torus $\mathbb{C}^g/\Gamma$, which in turn is equivalent to give an ample line bundle (note that this is not true for arbitrary Kähler manifolds.) A polarisation is then the datum of a Riemann form over the lattice Γ, or one of its equivalents.

According to Mumford[1] the study of complex algebraic curves is equivalent to the study of Riemann surfaces, and he doesn't fail to remark the perversity of algebraic geometers calling *curve* what analysts regard as a two dimensional object. In fact, the analytic point of view can sometimes be closer to intuition, as it is the case in what follows. To any compact Riemann surface it is possible to associate an Abelian variety, the *Jacobian variety*, the geometry of which closely reflects the geometry of the surface. "It is the existence of this auxiliary variety that makes the theory of compact manifolds of dimension one so much more beautiful and complete than the theory of complex manifolds of higher dimensions" [Clemens, 7]. The well known construction is based over two major theorems, due to Abel and Jacobi respectively, and deals with linear systems of divisors [see Lang, 32, for the abstract approach.]

Let C be a compact Riemann surface. Recall that C is covered by coordinate neighborhoods (U, z) where U can be identified with an open subset of $\mathbb{C}$ and z is the complex variable; if (U_1, z_1) is a second open set, then $z = u(z_1)$ and $z_1 = v(z)$ with u and v holomorphic functions on $U \cap U_1$. To give a differential form ω on C, one has to give an expression $f(z)dz$ on each (U, z) such that, on $U \cap U_1$,

$$f\big(u(z_1)\big) \cdot u'(z_1) \cdot dz_1 = f_1(z_1) \cdot dz_1$$

[1] See the monograph "Curves and their Jacobians," now available as appendix of the famous "red book" [42].

A differential form is *holomorphic* if each of the functions $f(z)$ is holomorphic (rather than meromorphic). Let ω be a differential on C and let γ be a path in $U \cap U_1$; then

$$\int_\gamma f(z)dz = \int_\gamma f_1(z_1)dz_1$$

Thus, it makes sense to integrate ω along any path in C. The sheaf of differential forms over C is called the *canonical sheaf* and it is denoted Ω^1.

Theorem 1.6. *The set of global sections of Ω^1 forms a g-dimensional vector space where g is the genus of C.*

We denote this vector space by $H^0(C, \Omega^1)$. If $\omega_1, \dots, \omega_g$ is a basis for the space, then every holomorphic differential is a linear combination, $\omega = \sum a_i \omega_i$, of the ω_i, and $\int_\gamma \omega = \sum a_i \int_\gamma \omega_i$; therefore it suffices to understand the finite set of integrals $\left\{ \int_\gamma \omega_1, \dots, \int_\gamma \omega_g \right\}$.

Recall (from topology) that C is a g-holed torus, and that $H_1(C, \mathbb{Z})$ has a canonical basis $\gamma_1, \dots, \gamma_{2g}$—roughly speaking each γ_i goes once round one hole. The vectors

$$\pi_j = \begin{pmatrix} \int_{\gamma_j} \omega_1 \\ \vdots \\ \int_{\gamma_j} \omega_g \end{pmatrix} \in \mathbb{C}^g, j = 1, \dots, 2g$$

are called the *period vectors*.

Theorem 1.7. *The $2g$ period vectors are linearly independent over $\mathbb{R}$.*

Let Γ denote the lattice generated by the period vectors, $\Gamma = \bigoplus \mathbb{Z}\pi_i$, then $\mathbb{C}^g/\Gamma$ is a complex torus. We shall see that in fact it is an abelian variety (i.e., it has a Riemann form). Fix a point P_0 on C. If P is a second point, and γ is a path from P_0 to P, then $\omega \mapsto \int_\gamma \omega$ is a linear map $H^0(C, \Omega^1) \to \mathbb{C}$. Note that if we replace γ with a different path γ' from P_0 to P, then γ' differs from γ by a loop. If the loop is contractible, then $\int_\gamma \omega = \int_{\gamma'} \omega$; otherwise the two integrals differ by a sum of periods.

Theorem 1.8 (Jacobi Inversion Formula)**.** *Let ℓ be a linear map $H^0(C, \Omega^1) \to \mathbb{C}$; then there exist points $P_1, \dots, P_g$ on C and paths γ_i from P_0 to P_i such that*

$$\ell(\omega) = \sum_i \int_{\gamma_i} \omega$$

for all $\omega \in H^0(C, \Omega^1)$.

Theorem 1.9 (Abel). *Let* $P_1, \ldots, P_r$ *and* $Q_1, \ldots, Q_r$ *be points on* C *(not necessarily distinct). Then there exists a meromorphic function* f *on* C *with poles exactly at the* P_i *and zeros exactly at the* Q_i *if and only if, for all paths* γ_i *from* P_0 *to* P_i *and all paths* ${\gamma_i}'$ *from* P_0 *to* Q_i*, there exists an element* $\gamma \in H_1(C, \mathbb{Z})$ *such that*

$$\sum_i \int_{\gamma_i} \omega - \sum_i \int_{\gamma_i'} \omega = \int_\gamma \omega$$

for all $\omega \in H^0(C, \Omega^1)$.

Let $\gamma \in H_1(C, \mathbb{Z})$; then $\omega \mapsto \int_\gamma \omega$ is a linear function on the vector space $H^0(C, \Omega^1)$, i.e. an element of $H^0(C, \Omega^1)^*$. Thus we have a map

$$H_1(C, \mathbb{Z}) \longrightarrow H^0(C, \Omega^1)^*$$

which is injective. Set

$$J(C) = H^0(C, \Omega^1)^* / H_1(C, \mathbb{Z})$$

The choice of a basis for $H^0(C, \Omega^1)$ identifies $J(C)$ with $\mathbb{C}^g/\Gamma$, thus $J(C)$ is a complex torus.

Theorem 1.10. *The intersection product*

$$H_1(C, \mathbb{Z}) \times H_1(C, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

is a Riemann form on $J(C)$*. Hence* $J(C)$ *is an abelian variety.*

Fix a point P on C. As we noted above, $\int_P^Q \omega$ doesn't make sense, because it depends on the choice of a path from P to Q. But two choices differ by a loop, and so $\omega \mapsto \int_P^Q \omega$ is well-defined as an element of

$$H^0(C, \Omega^1)^* / H_1(C, \mathbb{Z})$$

Thus we have a canonical map $\varphi \colon C \to J(C)$ sending P to 0.

Now consider the map

$$\begin{array}{ccc} \operatorname{Pic}^0 C & \longrightarrow & J(C) \\ \sum n_i P_i & \mapsto & \sum n_i \int_P^{P_i} \bullet \end{array}$$

The Jacobi inversion formula shows that this map is surjective (in fact it proves more than that). Abel's theorem shows that the kernel of the map is precisely the group of principal divisors. Therefore, the theorems of Abel and Jacobi show precisely that the above map defines an isomorphism

$$\operatorname{Pic}^0 C \longrightarrow J(C)$$

1.3 Prym Varieties

Let $\pi\colon \widetilde{C} \to C$ be an unramified double covering of a smooth complete algebraic curve C of positive genus g. Then by Riemann-Hurwitz relation we know that the genus of $\widetilde{C}$ is $2g-1$. If we denote by K_C the canonical class of C the argument is the following: since the covering is unbranched we have $\deg K_{\widetilde{C}} = 2 \deg K_C$, and for any curve C we have $\deg K_C = 2g - 2$. Let $J(\widetilde{C})$ and $J(C)$ be the Jacobians of the two curves and let $\iota\colon \widetilde{C} \to \widetilde{C}$ be the involution of $\widetilde{C}$ that exchanges the two sheets of the covering π. As Mumford [40] explains, ι acts on the Jacobian $J(\widetilde{C})$ of $\widetilde{C}$, and $J(\widetilde{C})$ splits under this action into a positive part and a negative part; we call the negative part P. A more analytical approach is described by Masiewicki [35]. The principal polarisation over $J(\widetilde{C})$ induces twice a principal polarisation on P, thus in particular P is an Abelian variety which we call the Prym variety of the double covering π. It's a fact that the positive part is given by $\pi^*\big(J(C)\big)$, therefore is g dimensional, thus the Prym variety P has dimension $g-1$.

Let $\mathcal{R}_5$ denote the moduli space of étale 2-sheeted coverings $\pi\colon \widetilde{C} \to C$, where C is a smooth complete curve of genus five. Later in the thesis (Theorem 3.14) we are going to prove that this space is *unirational*, meaning that there exists a dominant rational map $\mathbb{P}^n \dashrightarrow \mathcal{R}_5$ from a projective space to $\mathcal{R}_5$. By associating to any such double covering its Prym variety we construct a morphism

$$p_5 \colon \mathcal{R}_5 \longrightarrow \mathcal{A}_4$$

where $\mathcal{A}_4$ denotes the moduli space of Abelian varieties of dimension four, in a sense that will be made clear in §3.1. It is known [see Beauville, 4] that p_5 is generically surjective, and Donagi [13] gave a complete description of its two-dimensional fibres. Thus taking into account Theorem 3.14 the following result is proved.

Theorem 1.11. *$\mathcal{A}_4$ is unirational.*

This theorem is not original in that Clemens [8] proved it more than twenty years ago, but the argument is. Indeed Clemens didn't know that $\mathcal{R}_5$ is unirational and for his proof he relied on intermediate Jacobians.

Chapter 2

Quartic Surfaces and Double Covers of Curves

Given a quartic hypersurface in $\mathbb{P}^3$ with at least one ordinary double point, it is possible to construct a plane curve endowed with a double cover. The process involves building a quartic double solid, which turns out to be a conic bundle over $\mathbb{P}^2$; the plane curve will be then given by the discriminant of this conic bundle. We will ultimately be interested in the case of six double points: then the plane curve will be a five-nodal sextic, hence of genus five. Throughout the chapter we will work over a fixed field $\mathbb{K}$ of characteristic different from 2 and 3, which we will assume to be perfect.

2.1 Quartic Surfaces

Let X be a hypersurface in $\mathbb{P}^3$ of degree four, or equivalently let X be the projective spectrum of the graded algebra $S = R/(F)$, where F is an irreducible homogeneous polynomial of degree four inside $R = \mathbb{K}[x_0, \dots, x_3]$. A point $P \in X$ is *regular* if the ring $\mathscr{O}_{X,P} = S_{(\mathfrak{p})}$ is a regular local ring, otherwise P is *singular*, but it can be singular in many different ways: see Reid [45]. Without loss of generality we can assume P to be the point $[0 : 0 : 0 : 1]$ and restrict our attention to the affine open subset $D_h(x_3)$ of X, where the surface is given by $\operatorname{Spec} \mathbb{K}[x, y, z]/(f)$ with f an irreducible polynomial of degree four, and the point corresponds to the maximal ideal $\mathfrak{m} = (x, y, z)$. Writing f as the sum of its homogeneous components $f_1 + \cdots + f_4$, with subscripts indicating degree, properties of the polynomials f_d correspond to properties of the point P; for instance P is singular if and only if $f_1 = 0$, and P is said to be a *double point* when $f_1 = 0$ but $f_2 \neq 0$, which can be rephrased abstractly by saying that the local equation f inside the ring $\mathscr{O}_{X,P}$ defines a non-zero element of $\mathfrak{m}^2/\mathfrak{m}^3$. But even among double points only several different situations can arise, and one needs to look at the *tangent cone* to X at P for a finer distinction. This amounts to constructing the quotient of $\mathbb{K}[x, y, z]$ by the *leading form* of f (see Milne, [37]), which in this case is f_2, or to construct abstractly the associated graded ring of the local ring $\mathscr{O}_{X,P}$. When f_2 is an irreducible polynomial, or in other words when the associated graded ring is integral, we say that P is an *ordinary double point*. In terms of the graded algebra $R/(F)$ this situation is described by saying that F is a form of degree four contained in the ideal $(x_0, x_1, x_2)^2$, so that in particular F can be written as follows

$$F = u_2 x_3^2 + 2u_3 x_3 + u_4$$

where u_d is a form of degree d in $\mathbb{K}[x_0, x_1, x_2]$. The point P will then be an ordinary double point if the quadratic form u_2 is non-degenerate.

Consider now a quartic surface X with an isolated ordinary double point $P = [0 : 0 : 0 : 1]$, and let $\pi \colon X \setminus P \to \mathbb{P}^2$ be the projection with centre P. We can realise π abstractly as the morphism associated to the linear system over $\mathbb{P}^3$ of all the lines through P, which is given by the 3-dimensional linear space $H^0(\mathbb{P}^3, \mathscr{P}(1))$ where $\mathscr{P}$ is the ideal sheaf of the point, or more concretely by fixing a hyperplane H not containing P and identifying lines through P with points in H; either way we will construct the morphism of schemes induced by the homogeneous homomorphism

$$\mathbb{K}[y_0, y_1, y_2] \longrightarrow \mathbb{K}[x_0, \ldots, x_3]/(F)$$

sending y_i to x_i for $i = 0, 1, 2$. Any line in $\mathbb{P}^3$ through the point P intersects X in at most two other points therefore, according to standard terminology (see for example Liu, [33]), π is a quasi-finite morphism, meaning that each of its fibres is a finite set, but it is not finite. Nevertheless we can easily determine the maximal open subset over which it is finite.

Lemma 2.1. *The restriction of π to the open set $D_h(u_2) \subseteq \mathbb{P}^2$ is a finite morphism of degree two, and this is the maximal open subset over which π is finite.*

Proof. First observe that $X \setminus P$ is covered by the three affine open sets $D_h(x_i)$ for $i = 0, 1, 2$, and that locally π is induced by the inclusion of $\mathbb{K}[x, y]$ in the ring

$$A = \mathbb{K}[x, y][z]/(G_2 z^2 + 2G_3 z + G_4)$$

where G_d is the polynomial obtained from u_d by multiplication by $1/x_i^d$, in particular π is an affine morphism. Now it is enough to observe that for any multiplicatively closed subset S of $\mathbb{K}[x, y]$ the algebra $S^{-1}A$ is integral over $S^{-1}\mathbb{K}[x, y]$ if and only if $G_2 \in S$, and that over the field $\mathbb{K}(x, y)$ this is a vector space of dimension 2 generated by 1 and z. □

The locus of lines through P that are tangent to X away from P is the plane curve that we propose to study in this section. It is so closely related to the quartic surface that we will denote it by C_X. In fact what follows could be found already in Kreussler [30, Lemma 5.1].

Proposition 2.2. *C_X is a plane curve of degree six. More precisely, working in coordinates as above, it is given by the following projective spectrum*

$$C_X = \operatorname{Proj} \mathbb{K}[x_0, x_1, x_2]/(u_3^2 - u_2 u_4)$$

Furthermore if $Q \in X$ is a singular point (different from P), then the point $\pi(Q)$ is a singular point of the curve C_X.

Proof. In order to compute the equation of the curve we need in the first place to compute the fibre of π over any point of $\mathbb{P}^2$, and since it is an affine morphism we can restrict our attention to an open affine set. So for $i = 0, 1, 2$ consider the restriction of π to the open set $D_h(x_i) \subseteq \mathbb{P}^2$, which is induced by the inclusion of $\mathbb{K}[x, y]$ in the algebra A as above. A point of $\mathbb{P}^2$ is given over this open set by a maximal ideal $(x - a, y - b) \subseteq \mathbb{K}[x, y]$ and the fibre over this point is given by the tensor product, over $\mathbb{K}[x, y]$, of the quotient $\mathbb{K}[x, y]/(x - a, y - b)$ with A. In other words it is given by the spectrum of the ring $\mathbb{K}[x, y, z]/\mathfrak{a}$ where the ideal $\mathfrak{a}$ is generated by the polynomials $x - a$, $y - b$ and $G_2 z^2 + 2G_3 z + G_4$. This ring is naturally isomorphic to the quotient

$\mathbb{K}[z]/(g_2z^2 + 2g_3z + g_4)$ where g_d is the element of $\mathbb{K}$ obtained by computing $G_d(a,b)$. Now it is clear that the fibre over the point (a,b) consists of only one point if and only if either $g_2 = 0$ or $g_3^2 - g_2g_4 = 0$. But while in the former case we have one point with multiplicity one—the dimension of the vector space $\mathbb{K}[z]/(2g_3z + g_4)$ is one—in the latter we have multiplicity two as expected.

If $Q \in X$ is a double point (different from P) we can assume without loss of generality the point Q to be $[1 : 0 : 0 : 0]$, and concentrate our attention on the open set $D_h(x_0)$. Then, with the same notation as above, the polynomial $G_2z^2 + 2G_3z + G_4$ will be an element of $(x,y,z)^2$. This implies immediately that $G_4 \in (x,y)^2$ and that $G_3 \in (x,y)$, therefore $G_3^2 - G_2G_4 \in (x,y)^2$ which means $\pi(Q)$ is a singular point. □

It is perhaps useful to remark that the locus of points in $\mathbb{P}^2$ whose fibre under π consists of only one point is not irreducible. Indeed we have encountered in the proof above the two components given by the curve C_X and the conic $u_2 = 0$. We can say something more about this configuration, again by comparison with properties of the surface.

Corollary 2.3. *Let $\mathrm{Bl}_P(X) \to X$ be the blow-up of the surface X at the point P. Then the unique morphism of schemes $\mathrm{Bl}_P(X) \to \mathbb{P}^2$ that commutes with π is a double cover of $\mathbb{P}^2$ branched along the curve C_X. Moreover the image of the exceptional divisor in $\mathrm{Bl}_P(X)$ is the contact conic of C_X defined by the equation $u_2 = 0$.*

Proof. It is essentially enough to observe that the tangent cone of X at P is defined by the equation $u_2 = 0$ in $\mathbb{P}^3$. Indeed the blow-up of X in P is given by the proper transform of X in the blow-up of $\mathbb{P}^3$ at P (Eisenbud and Harris, [18], Proposition IV.21), whose exceptional locus is in correspondence with all the lines through P. Then the (-1)-curve inside $\mathrm{Bl}_P(X)$ will correspond to the set of lines in the tangent cone. To see that the conic $u_2 = 0$ is a contact conic of C_X simply look at the ideal $(u_2, u_3^2 - u_2u_4) = (u_2, u_3^2)$ inside the polynomial ring $\mathbb{K}[x_0, x_1, x_2]$. □

Let $Y_d = \operatorname{Proj} R/(u_d)$ be the cone over P defined by the form u_d, and let $Q \in X$ be any point different from P such that $\pi(Q) \in C_X$. Then we claim that the following holds.

Claim. $\pi(Q)$ is a singular point for C_X if and only if either Q is singular for X or $Q \in Y_2 \cap Y_3 \cap Y_4$.

Indeed if we denote by $\mathfrak{q}$ the homogeneous prime ideal inside $R/(F)$ corresponding to the point, then $Q \in X$ means $F \in \mathfrak{q}$ and $\pi(Q) \in C_X$ means $u_3^2 - u_2u_4 \in \mathfrak{q}$. Besides from the equality

$$u_2F = (u_2x_3 + u_3)^2 - (u_3^2 - u_2u_4)$$

we obtain immediately that also $u_2x_3 + u_3$ is in $\mathfrak{q}$. Now $\pi(Q)$ is a singular point if and only if $u_3^2 - u_2u_4 \in \mathfrak{q}^2$, and this happens if and only if $u_2F \in \mathfrak{q}^2$, which is equivalent to have either $u_2 \in \mathfrak{q}$ or $F \in \mathfrak{q}^2$. In the latter case Q is a singular point of X, in the former we also have $u_3 \in \mathfrak{q}$, since $u_2x_3 + u_3 \in \mathfrak{q}$, and $u_4 \in \mathfrak{q}$ since $2u_3x_3 + u_4 \in \mathfrak{q}$.

Proposition 2.4. *The generic quartic surface X with an isolated double point P gives rise to a curve C_X whose only singularities come from singularities of X. More precisely we require the equation $F = u_2x_3^2 + 2u_3x_3 + u_4$ to satisfy the condition*

$$\sqrt{(u_2, u_3, u_4)} = (x_0, x_1, x_2)$$

Proof. It is enough to rule out generically the case of a point Q contained in the intersection $Y_2 \cap Y_3 \cap Y_4$. Because of each Y_d is a cone over the point P we see immediately that if a point is in this intersection then the whole line through P and Q is contained in there. Thus we must rule out generically the case of a point in $\mathbb{P}^2$ contained in the intersection between a non-degenerate conic, a cubic and a quartic curve. Now observe that the intersection between the conic and a generic cubic consists of six points, and the generic quartic in the plane will not go through any of them. □

2.2 Double Solids and Conic Bundles

A *double solid* is a threefold that can be realised as a branched two-sheeted covering of $\mathbb{P}^3$. The definition was first introduced by Clemens [8] at a time when few specific examples of threefolds were known, in the attempt to develop an analogy with curve theory and clearly inspired by hyperelliptic curves. More recently Grooten [23] defines a double solid to be a closed subscheme Λ of the *weighted projective space* $\mathbb{P}(1,1,1,1,2)$, which in turn is defined to be the projective spectrum of the graded ring $\mathbb{K}[x_0,\ldots,x_3,y]$ where the degree of x_i is considered to be one, and the degree of y to be two. As Reid [46] explains, "Weighted projective spaces have appeared implicitly in algebraic geometry since ancient times; the most basic example is a hyperelliptic curve $y^2 = f_{2g+2}(x)$ viewed as a double cover of $\mathbb{P}^1$, that is a weighted hypersurface $C_{2g+2} \subset \mathbb{P}(1,1,g+1)$." The emphasis is on the double cover of $\mathbb{P}^3$, which is induced by the following inclusion of graded rings

$$\mathbb{K}[x_0,\ldots,x_3] \hookrightarrow \mathbb{K}[x_0,\ldots,x_3,y]/(y^2 - F)\ ,$$

and is branched along a hypersurface $X \subseteq \mathbb{P}^3$ of even degree, represented by the homogeneous polynomial F inside $R = \mathbb{K}[x_0,\ldots,x_3]$. For this reason the double solid given by $\operatorname{Proj}\left(\mathbb{K}[x_0,\ldots,x_3,y]/(y^2-F)\right)$ will be denoted here Λ_X. Observe that Λ_X is the union of the four open affine subsets $D_h(x_i)$ (where $i = 0,\ldots,3$), since $D_h(y) = D_h(y^2) = D_h(F)$ is contained in the union of these, so that the local equation of the threefold is in fact of the form $t^2 = f(x,y,z)$ near every point.

In the language of Clemens [8] the same construction looks very different and as such it deserves to be mentioned, but to do it we need first to recall some basic facts. It is a well known result that for any scheme X vector bundles over X and locally free sheaves on X are equivalent concepts, the equivalence being given by a natural one-to-one correspondence up to isomorphism. Indeed a *vector bundle* over X can be defined in this category simply as being a scheme E, endowed with a morphism $\pi\colon E \to X$, and isomorphic to $\mathbf{Spec}\,\mathrm{Sym}\,\mathscr{E}$ (as schemes over X), for some locally free sheaf $\mathscr{E}$ of rank n. Here $\mathbf{Spec}$ denotes the contravariant functor from quasi-coherent sheaves of $\mathscr{O}_X$-algebras to schemes over X that Eisenbud and Harris [18] call *global spec*. A *section* of a vector bundle over an open set U is then a morphism $s\colon U \to E$ such that $f \circ s$ is the open immersion of U in X, and the space of all sections is in fact a sheaf over X usually denoted $\mathscr{S}_{E/X}$.

Theorem 2.5. *Let X be a scheme. Then the two functors defined by $E \mapsto \mathscr{S}_{E/X}$ and* *by $\mathscr{E} \mapsto \mathbf{Spec}\,\mathrm{Sym}\,\mathscr{E}^\vee$ establish an equivalence of categories between vector bundles of rank n over X and locally free sheaves of rank n on X.*

Proof. It is enough to prove that the sheaf $\mathscr{S}_{E/X}$ is naturally identified with the locally free sheaf $\mathscr{E}^\vee$. To this purpose note that sections of π over the open subset U are in one-to-one correspondence with morphisms of quasi-coherent algebras $\mathrm{Sym}\,\mathscr{E}|_U \to \mathscr{O}_X|_U$, moreover if the open set U is also affine then by the adjoint property of **Spec** morphisms like these correspond to $\mathscr{O}_X(U)$-algebra homomorphisms $\mathrm{Sym}\,\mathscr{E}(U) \to \mathscr{O}_X(U)$. Now by the very definition of symmetric algebra there is another bijection with homomorphisms of $\mathscr{O}_X(U)$-modules between $\mathscr{E}(U)$ and $\mathscr{O}_X(U)$. □

Let now X be a hypersurface in $\mathbb{P}^3$ of even degree $2d$. Then X is defined by a global section s of the locally-free sheaf $\mathscr{O}(2d)$ on $\mathbb{P}^3$, which in turn can be considered as a morphism $s\colon \mathbb{P}^3 \to E_{2d}$ where E_{2d} is the line bundle $\mathbf{Spec}\,\mathrm{Sym}\,\mathscr{O}(-2d)$. Clemens [8] realises a double solid abstractly as the inverse image in E_d of the scheme theoretic image of s under the *squaring map* $E_d \to E_{2d}$. Over any open affine subset of the standard cover of $\mathbb{P}^3$, that is for any degree one monomial $e \in \{x_0, \dots, x_3\}$, this configuration is totally algebraic and corresponds to the following diagram

$$\begin{array}{ccc} S_{(e)}[\tau] & \xrightarrow[\tau \mapsto f]{\sigma} & S_{(e)} \\ {\scriptstyle q}\Big\downarrow {\scriptstyle \tau \mapsto t^2} & & \\ S_{(e)}[t] & & \end{array}$$

where f is the local equation of the hypersurface X, $\tau = 1/e^{2d}$ is a generator of the free module of rank one $\mathscr{O}_X(-2d)|_{D_h(e)}$, and $t = 1/e^d$ a generator for $\mathscr{O}_X(-d)|_{D_h(e)}$. The double solid is defined by the extension of $\ker\sigma$ via the homomorphism q, which is precisely the ideal $(t^2 - f)$ in accordance with the above discussion.

Observe that singularities in double solids occur only as singularities of the surface. Indeed among the partial derivatives of the local equation of Λ_X there is always the polynomial $2t$ which defines X inside the threefold. We are applying here the Jacobian criterion [Eisenbud, 15, §16.6] for which we need the base field to be a perfect field.

Quartic double solids correspond to hypersurfaces in $\mathbb{P}^3$ of degree four and are described by Clemens in detail first in [8] and later in [9]. Such threefolds are among the main characters of this thesis essentially because the singular ones are related to conic bundles over $\mathbb{P}^2$ or more generally over del Pezzo surfaces. So assume now X to be defined by a polynomial of degree four, and to contain at least one ordinary double point P_0. Denote by p the double cover of $\mathbb{P}^3$ given by Λ_X, and recall from Section 2.1 how to associate to X a plane curve C_X of degree six, in particular recall that projection from P_0 defines a morphism $\pi\colon X \setminus \{P_0\} \to \mathbb{P}^2$.

The main result of this section is that the blow-up $\mathrm{Bl}_{P_0}(\Lambda_X)$ of Λ_X at the point P_0 is in fact a conic bundle.

Proposition 2.6. *Let X be a quartic hypersurface in $\mathbb{P}^3$ with an ordinary double point P_0. Then the unique morphism of schemes f which makes the following diagram commute*

$$\begin{array}{ccccc} \mathrm{Bl}_{P_0}(\Lambda_X) & \longrightarrow & \Lambda_X & \xrightarrow{\;p\;} & \mathbb{P}^3 \\ & \searrow^{f} & & & \downarrow{\scriptstyle \pi} \\ & & & & \mathbb{P}^2 \end{array}$$

is a conic bundle over $\mathbb{P}^2$ for which the curve C_X is the locus of points whose fibre is a degenerate conic.

This statement was already contained in Clemens [8, p.222], but Kreussler [31, Theorem 2.1] improved it by giving an explicit description of all the morphisms involved. The argument starts with the construction of the conic bundle, which is realised as a divisor W_X inside a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$. Let $\mathscr{E}$ be the locally free sheaf of rank three $\mathscr{O} \oplus \mathscr{O}(-1) \oplus \mathscr{O}(-2)$ over $\mathbb{P}^2$, and construct $\mathbb{P}(\mathscr{E})$ by taking global Proj of the symmetric powers of $\mathscr{E}$

$$\mathbb{P}(\mathscr{E}) = \mathbf{Proj}\,\mathrm{Sym}\,\mathscr{E}$$

as described for instance in Hartshorne [28, §II.7]. In order to define the divisor we are interested in we take constant non-zero global sections $z_0 \in H^0(\mathbb{P}^2, \mathscr{E})$, $z_1 \in H^0(\mathbb{P}^2, \mathscr{E}(1))$ and $z_2 \in H^0(\mathbb{P}^2, \mathscr{E}(2))$, so that for any open set $U \subseteq \mathbb{P}^2$ any element $s \in \mathscr{E}(U)$ can be written in a unique way as a sum

$$s = z_0|_U s_0 + z_1|_U s_1 + z_2|_U s_2$$

with $s_d \in H^0(U, \mathscr{O}(-d)|_U)$. In particular over any open set of the standard cover of the projective plane, $D_h(x_i) \subseteq \operatorname{Proj} S$ with $S = \mathbb{K}[x_0, x_1, x_2]$, the restriction of $\mathscr{E}$ is a coherent sheaf given by

$$\begin{aligned}\mathscr{E}|_{D_h(x_i)} &= \mathscr{O}|_{D_h(x_i)} \oplus \mathscr{O}(-1)|_{D_h(x_i)} \oplus \mathscr{O}(-2)|_{D_h(x_i)} \\ &= \operatorname{Shf} S_{(x_i)} \oplus \operatorname{Shf} \frac{1}{x_i} S_{(x_i)} \oplus \operatorname{Shf} \frac{1}{x_i^2} S_{(x_i)} \\ &= \operatorname{Shf} \left(S_{(x_i)} \oplus \frac{1}{x_i} S_{(x_i)} \oplus \frac{1}{x_i^2} S_{(x_i)} \right)\end{aligned}$$

thus a local base for the free $S_{(x_i)}$-module $\mathscr{E}(D_h(x_i))$ is realised by

$$z_0, \ \frac{z_1}{x_i}, \ \frac{z_2}{x_i^2}$$

and taking symmetric powers is equivalent to forming the polynomial ring over $S_{(x_i)}$ in these variables. We now define the divisor W_X inside $\mathbb{P}(\mathscr{E})$ by means of the following polynomial expression inside $H^0(\mathbb{P}^2, \mathscr{O}(4) \otimes \operatorname{Sym}^2 \mathscr{E})$

$$-z_2^2 + z_1^2 u_2 + 2 z_1 z_0 u_3 + z_0^2 u_4$$

By this we mean that on any $D_h(x_i)$ the conic bundle is given by the projective spectrum of the following graded algebra over $\mathbb{K}[x, y]$

$$\mathbb{K}[x, y][\xi_0, \xi_1, \xi_2] / (-\xi_2^2 + \xi_1^2 G_2 + 2\xi_1 \xi_0 G_3 + \xi_0^2 G_4)$$

where ξ_0, ξ_1, ξ_2 is a local base for $\mathscr{E}$ and G_d is the polynomial obtained from u_d by multiplication by $1/x_i^d$ as in Lemma 2.1.

To complete the argument Kreussler [30] defines a morphism $\Phi\colon W_X \to \mathbb{P}^3$ and proves that it factors through p: then he observes that W_X endowed with this morphism to Λ_X is actually the blow-up of the double solid and shows the compatibility with π.

2.3 How genus five curves arise

As we did before we begin this section by considering a quartic hypersurface in $\mathbb{P}^3$, but here we will assume the singularities of X to consist precisely of six isolated ordinary double points, $P_0, \ldots, P_5$. With respect to P_0 we will also assume X to be generic, in the precise sense of Proposition 2.4, and construct the plane curve C_X as in Section 2.1. The first observation is that under these hypotheses the curve C_X is a plane sextic with precisely five nodes. Thus using the well known formula for plane curves $g = (d-1)(d-2)/2 - \delta$ where δ is the number of nodes [see Arbarello et al., 1], we conclude immediately that the genus of C_X is five. Next we resolve the singularities of the curve by blowing up the plane at the five points $P_0, \ldots, P_5$.

Proposition 2.7. *Let C be a plane curve of degree six with five ordinary nodes and let $\widetilde{\mathbb{P}^2}$ be the blow-up of the projective plane in the five points corresponding to nodes of C. Then the strict transform of C inside $\widetilde{\mathbb{P}^2}$ is a canonically embedded non-singular curve.*

Proof. According to Griffiths and Harris [22, §4.4], the blow-up of $\mathbb{P}^2$ in five points is a Del Pezzo surface embedded in $\mathbb{P}^4$ as the complete intersection of two quadrics by the complete linear system $|-K_{\widetilde{\mathbb{P}^2}}|$, where $K_{\widetilde{\mathbb{P}^2}}$ is a canonical divisor in $\widetilde{\mathbb{P}^2}$. Thus the strict transform $\widetilde{C}$ of C will be embedded in $\mathbb{P}^4$ by the restriction of this linear system. Now let H be a line in $\mathbb{P}^2$, let $E_1, \ldots, E_5$ be the exceptional divisors in S and let σ be the blow-up morphism.

$$\begin{aligned}
\left|-K_{\widetilde{\mathbb{P}^2}}\right| &= \left|\sigma^*(3H) - \sum_{i=1}^{5} E_i\right| = \left|\sigma^*(-3H) + \sigma^*(6H) - \sum_{i=1}^{5} E_i\right| \\
&= \left|\sigma^*(-3H) + \sigma^*(C) - \sum_{i=1}^{5} E_i\right| \\
&= \left|\sigma^*(-3H) + \widetilde{C} + 2\sum_{i=1}^{5} E_i - \sum_{i=1}^{5} E_i\right| \\
&= \left|\sigma^*(-3H) + \sum_{i=1}^{5} E_i + \widetilde{C}\right| \\
&= \left|K_{\widetilde{\mathbb{P}^2}} + \widetilde{C}\right|.
\end{aligned}$$

Therefore the curve $\widetilde{C}$ is embedded in $\mathbb{P}^4$ by the restriction of the linear system $|K_{\widetilde{\mathbb{P}^2}} + \widetilde{C}|$, but by the adjunction formula this is just the canonical linear system of the curve. Observe also that $\widetilde{C}$ is non-singular because the singularities of C were only ordinary nodes. □

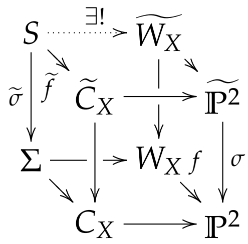
By Proposition 2.6 the blow-up of the double solid Λ_X at the point P_0 is a conic bundle over $\mathbb{P}^2$ branched along the curve C_X. Exploiting the existence of this conic bundle we can now define in a natural way an unbranched double cover of the normalisation of C_X.

Proposition 2.8. *Let X be a generic quartic hypersurface in $\mathbb{P}^3$ with six isolated ordinary double points $P_0, \ldots, P_5$, and let $\widetilde{C}_X$ be the canonical model of the associated plane sextic with respect to P_0. Then there exists a natural unbranched double cover of the curve $\widetilde{C}_X$.*

Proof. Let W_X be the blow-up of the double solid Λ_X at the point P_0 viewed as a conic bundle over $\mathbb{P}^2$, and let $\widetilde{W_X}$ be the pull-back of W_X along the blow-up of $\mathbb{P}^2$ at the five points $P_1, \ldots, P_5$ corresponding to the singularities of the curve C_X. In other words consider the following fibred product diagram

$$\begin{array}{ccc} \widetilde{W_X} & \longrightarrow & \widetilde{\mathbb{P}^2} \\ \downarrow & & \downarrow \sigma \\ W_X & \xrightarrow{f} & \mathbb{P}^2 \end{array}$$

By Proposition 2.7, the strict transform of the curve C_X inside $\widetilde{\mathbb{P}^2}$ is the canonical model $\widetilde{C}_X$. Abstractly we can realise $\widetilde{C}_X$ by blowing up C_X alone, the process is described for instance in Hartshorne [28, §II.7] or in Eisenbud and Harris [18, Proposition IV-21]. Let Σ be the preimage of the curve C_X inside the threefold W_X and consider also the blow up of Σ along the preimage of the five points $P_1, \ldots, P_5$, which we call S. The situation is summarised by the following commutative cubic diagram: the base is the fibered product defining Σ as a closed subscheme of W_X; the three "visible" vertical arrows are blow-ups and the vertical face whose base if f is the fibred product above. Note incidentally the existence and uniqueness of the dotted arrow, guaranteed by the universal property of the fibred diagram just mentioned. It would be interesting to compare the top face of this cube with a fibred product, or in other words to compare S with the preimage of $\widetilde{C}_X$ inside $\widetilde{W_X}$.



Let $\nu\colon \widetilde{S} \to S$ be the normalisation of S, and consider the Stein factorisation [Hartshorne, 28, Corollary III.11.5] of the morphism $\widetilde{f} \circ \nu$ to $\widetilde{C}_X$

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\nu} & S \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ \Gamma & \xrightarrow{g} & \widetilde{C}_X \end{array}$$

where f' is a projective morphism with connected fibres and g is a finite morphism. Evidently the degree of g is two, as the degenerate conic consists of two lines.

The double cover g is unbranched simply because the restriction of the conic bundle W_X to the curve C_X consists of a fibration by pairs of distinct lines. To see this recall from the previous section the local equation for the conic bundle, and construct the preimage of the point P of $\mathbb{P}^2$ given by the maximal ideal $(x-a, y-b) \subseteq \mathbb{K}[x,y]$, which will be given by the form

$$-\xi_2^2 + \xi_1^2 g_2 + 2\xi_1\xi_0 g_3 + \xi_0^2 g_4$$

where g_d is the element of $\mathbb{K}$ obtained by computing $G_d(a,b)$ as in Proposition 2.2. The associated matrix is then the following

$$\begin{pmatrix} g_4 & g_3 & 0 \\ g_3 & g_2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Taking into account that we are assuming g_2, g_3 and g_4 never to be simultaneously null, this matrix will always have rank two, thus the quadratic polynomial will always consist of two distinct lines. □

It is still possible that the double cover g might be trivial, that is, it might consist of two copies of the curve $\widetilde{C}_X$. Observe that this will be the case if and only if Γ is a disconnected curve, therefore reducible; but if Γ is reducible then $\widetilde{S}$ is also reducible. Taking into account that $\widetilde{S}$ is obtained from S by normalisation we see now that for the double cover to be non-trivial it is enough to show that S is irreducible, and since S is constructed by blowing up Σ it will be irreducible if and only if Σ is.

Going back again to the local equations of the conic bundle we are now interested in the divisor inside W_X defined by the equation $u_3^2 - u_2u_4$, which over the open set $D_h(x_i)$ inside $\mathbb{P}^2$ is given by the following ideal in $\mathbb{K}[x,y][\xi_0, \xi_1, \xi_2]$

$$(-\xi_2^2 + \xi_1^2 G_2 + 2\xi_1\xi_0 G_3 + \xi_0^2 G_4, G_3^2 - G_2G_4)$$

where G_d is the polynomial obtained from u_d by multiplication by $1/x_i^d$. Note that in order to prove that Σ is irreducible we need to show that the ideal above is prime over any of the three standard open sets that cover $\mathbb{P}^2$, in other words we have to show that any of three ideals is prime.

If we denote by $\mathcal{H}$ the Hilbert scheme of hypersurfaces in $\mathbb{P}^3$ of degree four, then these ideals will be prime for X contained in some open subset of $\mathcal{H}$. We need only to check that this is not empty, then we shall have constructed a connected unbranched double cover of C_X for a general quartic surface X. The relevant computation was carried out with *Macaulay 2* and details can be found in Section 4.1.

2.4 An Explicit Example

Our purpose is now to compute an explicit example of a quartic hypersurface $\mathbb{X}$ in $\mathbb{P}^3$ with six ordinary double points. We will require $\mathbb{X}$ to be generic in the sense of Proposition 2.4, and perform over it all the constructions described so far. Furthermore if later in the thesis we again require to specialise our arguments we want this to be our standard test example.

For ease of computation we work over the finite field $\mathbb{F}_{101}$ (the prime 101 has no special importance here.) This is harmless in that if a variety in characteristic 0 is reducible so is its reduction modulo a prime p. Observe also that we shall have no restrictions in performing usual operations such as for instance computing derivatives to define the singular locus. Indeed the Jacobian criterion [Eisenbud 15, §16.6] holds over any perfect field and according to Zariski and Samuel [55, §II.4] these include $\mathbb{F}_p$ for any prime p, as well as any algebraically closed field. All the computations described in this thesis were performed with *Macaulay 2* [21].

As usual we realise $\mathbb{P}^3$ as the projective spectrum of the polynomial ring $R = \mathbb{K}[x_0, \ldots, x_3]$ and we fix the points $P_0, \ldots, P_5$ corresponding to the following prime ideals

$$\begin{aligned}
&\mathfrak{p}_0 = (x_0, x_1, x_2), \quad \mathfrak{p}_1 = (x_0, x_1, x_3), \quad \mathfrak{p}_2 = (x_0, x_2, x_3), \\
&\mathfrak{p}_3 = (x_1, x_2, x_3), \quad \mathfrak{p}_4 = (x_0 - x_1, x_0 - x_2, x_0 - x_3), \\
&\mathfrak{p}_5 = (2x_0 - x_1, 3x_0 - x_2, 4x_0 - x_3)
\end{aligned}$$

The surface $\mathbb{X}$ is then defined by a single irreducible homogeneous polynomial F of degree four contained in the intersection $I = \bigcap_{i=0}^{5} \mathfrak{p}_i^2$, and as such it can be randomly selected. Thus we instruct the program to compute a system of generators for the ideal I, which will be returned in the form of the matrix associated to the homogeneous linear morphism whose image is the ideal I

$$a \colon \bigoplus_{i=1}^{11} R(-d_i) \longrightarrow R$$

where d_i are the degrees of the generators. So the next step is to compute a random matrix of homogeneous polynomials in R such that when multiplied with the above the result is a form of degree four, or in other words a linear morphism

$$b \colon R(-4) \longrightarrow \bigoplus_{i=1}^{11} R(-d_i)$$

selected randomly for the composition $a \circ b$ to be defined by a single polynomial of degree four F.

The result must now be tested, firstly to ensure its singular locus amounts to just the six double points and then to ensure it verifies the generic condition described in Proposition 2.4. The first is easily done by computing the derivatives of the polynomial F and asking for the Hilbert polynomial of the variety defined by $(F, \partial F/\partial x_0, \dots, \partial F/\partial x_3)$ to be constant and equal to 6. For the second we need to write F in the form $u_2 x_3^2 + 2u_3 x_3 + u_4$ and compute the Hilbert polynomial of (u_2, u_3, u_4). Since these varieties are cones over P_0 the condition is satisfied if they meet in P_0 only, so we expect the Hilbert polynomial to be constant (in fact equal to 24). After these tests it appeared the following homogeneous forms satisfy our requirements.

$$\begin{aligned}
u_2 =& 19x_0^2 - 33x_0x_1 + 50x_1^2 - 13x_0x_2 + 50x_1x_2 - 15x_2^2 \\
u_3 =& -2x_0^2x_1 - 35x_0x_1^2 - 18x_0^2x_2 - 8x_0x_1x_2 - 36x_1^2x_2 - 4x_0x_2^2 + 45x_1x_2^2 \\
u_4 =& -38x_0^2x_1^2 - 32x_0^2x_1x_2 - 32x_0x_1^2x_2 - 6x_0^2x_2^2 - 38x_0x_1x_2^2 + 2x_1^2x_2^2
\end{aligned}$$

Now we are left with the plane curve $u_3^2 - u_2u_4 = 0$, which we know has five ordinary nodes, and we want to compute its normalisation. In other words we want to compute the blow-up of the plane in five points, and to do so we can follow Griffiths and Harris [22, §4.4] and consider the linear system of cubic curves passing through the points. Details of this computation are described in §4.1, here we are more interested in the result which is the ideal in $\mathbb{P}^4$ generated by the following three polynomials.

$$\begin{aligned}
F_1 =& y_1y_2 + y_2^2 - y_0y_3 - 34y_2y_3 + 33y_2y_4 \\
F_2 =& y_0y_2 - 2y_2^2 - 33y_2y_3 - y_1y_4 + 35y_2y_4 \\
F_3 =& y_0^2 - 48y_0y_1 + 5y_1^2 + 12y_2^2 + 48y_0y_3 + 31y_1y_3 + 4y_2y_3 - 20y_3^2 + 42y_0y_4 + \\
& - 50y_1y_4 - 45y_2y_4 + 17y_3y_4 - 19y_4^2
\end{aligned}$$

Thus the normalisation of the curve C_X is given by the intersection of three quadrics in $\mathbb{P}^4$, and is therefore canonically embedded. Note that this also proves that the singularities of the original plane curve are at worst nodes: indeed they are resolved by a single process of blow-up.

Chapter 3

Building Families

We are now in a position to describe a rational map ϱ from the space $\mathcal{Q}$ of quartic hypersurfaces in $\mathbb{P}^3$ with six isolated singularities to $\mathcal{R}_5$, the space of étale double covers of curves of genus five. Essentially this amounts to defining a natural way to construct families of curves from families of surfaces, but to make this precise we will need to use the formalism of category theory. Besides we will also show that $\mathcal{Q}$ is *rational*, meaning that there exists a dominant birational map from a projective space $\mathbb{P}^n$ to $\mathcal{Q}$, thus paving the way for the following result from which this thesis takes its title

Theorem (3.14). *The moduli space $\mathcal{R}_5$ of étale two-sheeted coverings of curves of genus five is unirational.*

Unirationality is a weaker condition than rationality, in that it requires only the existence of a dominant rational map from $\mathbb{P}^n$. Therefore, on the last step in the proof of the theorem it will be sufficient to prove that ϱ is a dominant map. For simplicity we will always assume our schemes to be of finite type over $\mathbb{C}$.

3.1 Moduli Problems and Functors

A well known and widespread point of view is to identify a scheme X with its *functor of points*, which is defined by associating to any scheme Y the set of morphisms $h_X(Y) = \mathrm{Mor}(Y, X)$ and named from the observation that $h_X(\mathrm{Spec}\, K)$ is the set of *K-rational points* of X, for any field K. The construction is explained in detail by Eisenbud and Harris [18] and consists of defining a fully faithful functor

$$h\colon\ (\mathit{Schemes}) \longrightarrow \operatorname{Fun}\left((\mathit{Schemes})^{\circ}, (\mathit{Sets})\right)$$

from the category of schemes to the category of (contravariant) functors between schemes and sets, whose arrows are natural transformations [see for instance Mac Lane, 34, §II.4]. Thus any morphism $u\colon X \to X'$ gives rise by composition to a natural transformation $h(u)$ between the functors h_X and $h_{X'}$, with naturality square given by the following diagram

$$\begin{array}{ccccc} Z & & h_X(Z) & \xrightarrow{h(u)_Z} & h_{X'}(Z) \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle h_X(\alpha)}\downarrow & & \downarrow{\scriptstyle h_{X'}(\alpha)} \\ Y & & h_X(Y) & \xrightarrow{h(u)_Y} & h_{X'}(Y) \end{array}$$

In fact the construction of h makes sense in any category, not just $(\mathit{Schemes})$, and it is a basic result in category theory that it always gives rise to a fully faithful functor. The following is the precise statement according to Eisenbud and Harris [18, Lemma VI-1].

Lemma 3.1 (Yoneda). *Let $\mathscr{C}$ be a category and let X, X' be objects of $\mathscr{C}$.*

- (a) *If F is any contravariant functor from $\mathscr{C}$ to the category of sets, the natural transformations from $\operatorname{Mor}(-, X)$ to F are in natural correspondence with the elements of $F(X)$*

$$\operatorname{Nat}\left(\operatorname{Mor}(-, X), F\right) \longleftrightarrow F(X)$$

- (b) *If the functors $\operatorname{Mor}(-, X)$ and $\operatorname{Mor}(-, X')$ from $\mathscr{C}$ to the category of sets are isomorphic, then $X \cong X'$. More generally, the maps of functors from $\operatorname{Mor}(-, X)$ to $\operatorname{Mor}(-, X')$ are the same as maps from X to X'*

$$\operatorname{Nat}\left(\operatorname{Mor}(-, X), \operatorname{Mor}(-, X')\right) \longleftrightarrow \operatorname{Mor}(X, X'),$$

that is, the functor $h\colon \mathscr{C} \to \operatorname{Fun}(\mathscr{C}^{\circ}, (\mathit{Sets}))$ sending X to h_X is fully faithful.

Posing a moduli problem amounts to defining a (contravariant) functor F from the category of schemes to (Sets) by specifying what *families* of objects to consider. In general, for any scheme B, a family will be a morphism $f\colon \mathscr{X} \to B$

with some properties depending on the particular problem, but always with fibres required to be of a certain kind. To any morphism of schemes $B \to B'$ a moduli functor will then associate an arrow defined by pulling-back families or in other words by taking fibred products. Given a moduli functor, or also given any $F\colon (Schemes)^\circ \to (Sets)$, it is now natural to ask whether it is of the form h_X for some scheme X or not, and if it is to give it a special name. In this case we say that F is *representable* by X or that X *represents* F.

Definition. If F is representable by $\mathcal{M}$, then we say that the scheme $\mathcal{M}$ is a *fine moduli space* for the moduli problem F.

The classic example of a fine moduli space is the Hilbert scheme, parametrising closed subschemes of $\mathbb{P}^r$ with fixed Hilbert polynomial $P = P(m)$. The moduli problem is first presented formally by defining the functor $H_{P,r}$, whose value on B is the set of proper flat families

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{i} & \mathbb{P}^r \times B \\ & \searrow^{\varphi} & \downarrow \\ & & B \end{array}$$

with $\mathscr{X}$ having Hilbert polynomial P, and i being a closed immersion. The main result, due to Grothendieck [24], is then the following.

Theorem 3.2 (Grothendieck). *$H_{P,r}$ is representable by a projective scheme $\mathcal{H}_{P,r}$.*

Fine moduli spaces are characterised by the existence of a *universal family*, that is a family $\mathscr{U} \to \mathcal{M}$ from which every other one can be obtained. To see this observe that the functor F is representable by $\mathcal{M}$ if and only if morphisms in $h_\mathcal{M}(B)$ are in bijection with families in $F(B)$. Thus a family over B is the same as a morphism $\alpha \in h_\mathcal{M}(B)$, which in turn is given by the image of the identity under $h_\mathcal{M}(\alpha)$.

$$\begin{array}{ccccc} \mathcal{M} & \quad & F(\mathcal{M}) & \longleftrightarrow & h_\mathcal{M}(\mathcal{M}) \\ \uparrow{\scriptstyle \alpha} & & {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle h_\mathcal{M}(\alpha)} \\ B & & F(B) & \longleftrightarrow & h_\mathcal{M}(B) \end{array}$$

The universal family is then the unique element in $F(\mathcal{M})$ corresponding to the identity of $\mathcal{M}$.

As an example consider the moduli space of hypersurfaces in $\mathbb{P}^r$ of degree d. If X is a projective variety contained in $\mathbb{P}^r$ then it is well known (see for instance Harris and Morrison [26], §1.B) that X is a hypersurface of degree d if and only if its Hilbert polynomial is the following

$$P_X(m) = \binom{r+m}{r} - \binom{r+m-d}{r}$$

Thus this moduli space is in fact the Hilbert scheme $\mathcal{H}_{P,r}$, and it is given by the projective space $\mathbb{P}(V)$, where V is the vector space of homogeneous polynomials of degree d in $r+1$ variables. The universal family is the closed subset of the product $\mathbb{P}(V) \times \mathbb{P}^r$ defined by the equations $\{(F, P) \mid F(P) = 0\}$.

Unfortunately, the existence of a universal family is a requirement too restrictive for many natural moduli problems; in other words there are examples of moduli functors which are not representable but nevertheless describe interesting classes of objects. Perhaps the most famous of these is the functor describing families of smooth projective curves of genus g. One way of getting around this problem is to look for the existence of a scheme $\mathcal{M}$ that "best represents" the functor, satisfying a weaker universal property.

Definition. A scheme $\mathcal{M}$ and a natural transformation Φ from F to the functor of points of $\mathcal{M}$ are a *coarse moduli space* for the moduli problem F if

1) The map $\Phi_{\mathbb{K}}\colon F(\operatorname{Spec}\mathbb{K}) \to h_{\mathcal{M}}(\operatorname{Spec}\mathbb{K})$ is a set bijection;

2) For any scheme $\mathcal{M}'$ and any natural transformation $\Psi\colon F \to h_{\mathcal{M}'}$, there is a unique morphism $u\colon \mathcal{M} \to \mathcal{M}'$ such that $\Psi = h(u) \circ \Phi$.

The moduli functor $\mathbf{m}_g$ is defined by considering smooth proper morphisms $\mathscr{X} \to B$ whose fibres are nonsingular irreducible projective curves of genus g.

Theorem 3.3. *Let $g \geq 2$. The functor $\mathbf{m}_g$ is coarsely represented by a quasi-projective scheme $\mathcal{M}_g$, irreducible and $(3g-3)$-dimensional.*

The problem was first considered in the complex case by Riemann [47], who proved that the space of non isomorphic complex structures (named by him "moduli") over a compact connected surface of genus $g \geq 2$ has complex dimension $3g - 3$. However in this generality the result is explained in detail by, and mostly due to, Deligne and Mumford [11].

Lemma 3.4. *Let $\mathcal{F}$ be a fine moduli space for the moduli problem* $\mathbf{F}$*, and let $\mathcal{M}$ be a coarse moduli space for the moduli problem* $\mathbf{m}$*. Then to give a morphism of schemes $\mathcal{F} \to \mathcal{M}$ it is enough to give a natural transformation $\mathbf{F} \to \mathbf{m}$.*

Proof. This is just an application of Yoneda's Lemma (3.1). Indeed by definition there are natural transformations $\mathbf{m} \to h_{\mathcal{M}}$ and $\mathbf{F} \to h_{\mathcal{F}}$, the last of which is assumed to be invertible, so composition with another one from $\mathbf{F}$ to $\mathbf{m}$ will define a natural transformation between the functors $h_{\mathcal{F}}$ and $h_{\mathcal{M}}$. □

This lemma is used in practice to define rational maps between moduli spaces. Indeed what generally happens is that there exists a Zariski open subset of a coarse moduli space over which the moduli functor will become representable. This is the case for instance of curves, where the open set is given by the locus of curves without automorphisms.

Consider now the functor $\mathbf{r}_g$ ($g \geq 2$) defined over the scheme B by considering families of smooth projective curves of genus g with a connected étale double cover. According to Beauville [3, §6] this functor is coarsely represented by an irreducible quasi-projective scheme $\mathcal{R}_5$. The dimension of this moduli space is $3g - 3$, the same as the dimension of $\mathcal{M}_5$. To see this recall that double covers of curves correspond one to one to non-trivial line bundles $\mathscr{L}$ whose square $\mathscr{L} \otimes \mathscr{L}$ is trivial (compare Hartshorne [28], Exercise IV.2.7), so the families considered by the functor $\mathbf{r}_g$ are just families of curves with an extra structure. Such a line bundle on a smooth irreducible projective curve C of genus g corresponds to one of the $2^{2g} - 1$ nontrivial 2-torsion points in the Jacobian of C. So forgetting about the extra information we clearly define a natural transformation between $\mathbf{r}_g$ and $\mathbf{m}_g$, and applying Lemma 3.4 we then obtain a (at least rational) map $\mathcal{R}_g \to \mathcal{M}_g$ with finite fibres.

3.2 The Moduli Space of Quartic Surfaces

Let $\mathcal{Q}$ be the moduli space of "quartic hypersurfaces in $\mathbb{P}^3$ with six isolated ordinary double points, one of which is marked, up to projective equivalence."

As described in Section 3.1 this is equivalent to considering the moduli functor **Q** whose value on the scheme B is the set of proper flat families

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{i} & \mathbb{P}^r \times B \\ & \searrow^{\varphi} & \downarrow \\ & & B \end{array}$$

where i is a closed immersion and for every $b \in B$ the fibre of φ over the point b is a quartic hypersurface as described above, but over the residue field $K(b) = \mathscr{O}_{B,b}/\mathfrak{m}_b$. Our first goal is to prove that this is in fact a fine moduli space, or in other words that this functor is representable by a scheme $\mathcal{Q}$.

Proposition 3.5. *$\mathcal{Q}$ is a fine moduli space. More precisely it is a locally closed subscheme of the Hilbert scheme of quartic hypersurfaces in $\mathbb{P}^3$.*

Proof. To begin with consider the Hilbert scheme of quartic hypersurfaces in $\mathbb{P}^3$. This is the fine moduli space of projective varieties contained in $\mathbb{P}^3$ with Hilbert polynomial

$$P_X(m) = \binom{3+m}{3} - \binom{m-1}{3} = 2m^2 + 2$$

and can be realised as $\mathbb{P}(R_4)$, where R is the polynomial ring $\mathbb{K}[x_0, \ldots, x_3]$, or in other words as $\mathbb{P}\big(H^0(\mathbb{P}^3, \mathscr{O}(4))\big)$.

However we are interested in surfaces up to projective equivalence, so to take into account the action of $PGL(4)$ we fix the five points in $\mathbb{P}^3$ corresponding to the following prime ideals

$$\mathfrak{p}_0 = (x_0, x_1, x_2), \ \ \mathfrak{p}_1 = (x_0, x_1, x_3), \ \ \mathfrak{p}_2 = (x_0, x_2, x_3),$$
$$\mathfrak{p}_3 = (x_1, x_2, x_3), \ \ \mathfrak{p}_4 = (x_0 - x_1, x_0 - x_2, x_0 - x_3)$$

So we are now looking at "quartic hypersurfaces in $\mathbb{P}^3$ with five isolated ordinary double points, up to projective equivalence." This is again a fine moduli space, in that it is the closed subscheme of the Hilbert scheme given by $\mathbb{P}(I_4)$ where $I = \bigcap_{i=0}^4 \mathfrak{p}_i^2$ is the ideal of the five double points. In other words we are looking at the linear space $H^0(\mathbb{P}^3, \mathscr{I}(4))$, where we denote $\mathscr{I}$ the sheaf of ideals defined by I, inside $H^0(\mathbb{P}^3, \mathscr{O}(4))$, and then realising a projective space with it. Observe that the dimension $h^0(\mathbb{P}^3, \mathscr{I}(4))$ gives the value of the Hilbert function of $\mathscr{I}$ in degree four, which a direct computation shows to be 15.

Going on, we ask now for a sixth double point. To realise this we can take the product $\mathbb{P}(I_4) \times \mathbb{P}^3$ and consider the closed subscheme B_0 given by the relations

$$\left\{(F, \mathfrak{p}) \mid F \in \mathfrak{p}^2\right\} = \left\{(F, P) \;\middle|\; F(P) = \frac{\partial F}{\partial x_i}(P) = 0\right\}$$

Projecting onto $\mathbb{P}(I_4)$ we consider now the scheme theoretic image of B_0 (see §1.1) which is a closed subscheme B of $\mathbb{P}(I_4)$, and observe that in the universal factorisation

$$\begin{array}{ccc} B_0 & \xrightarrow{\beta} & B \\ & \searrow & \downarrow \\ & & \mathbb{P}(I_4) \end{array}$$

the dominant morphism β is also proper, because it is the external morphism of a composition which is proper [Hartshorne, 28, Exercise II.4.8]. As a consequence β is surjective and the scheme theoretic image of B_0 coincides with the set theoretic image. Observe also that B is irreducible by Corollary 1.4. However B contains all the possible degenerations of a quartic hypersurface with six double points, while we are interested in those hypersurfaces with no other singularities. As this is clearly an open condition we have proved that $\mathcal{Q}$ is an open subset of an irreducible closed subset of the Hilbert scheme, and thus it is a fine moduli space. □

The proof above in fact reveals something more about $\mathcal{Q}$, namely we now know its dimension and most of all we know that it is a *unirational* space. The last property in particular is of primary importance for the purposes of this thesis.

Corollary 3.6. *$\mathcal{Q}$ is a* 13*-dimensional, unirational variety.*

Proof. We have seen in the proof of the Proposition above that there is a dominant rational morphism from a closed subscheme B_0 of the product $\mathbb{P}(I_4) \times \mathbb{P}^3$ to $\mathcal{Q}$, given by the projection onto $\mathbb{P}(I_4)$. Now focus on the other projection and observe that a generic point $P \in \mathbb{P}^3$ defines four independent conditions on the linear space $H^0(\mathbb{P}^3, \mathscr{I}(4))$, thus there exists an open subset U of $\mathbb{P}^3$

over which these conditions define a vector bundle E of rank 11. To see that U is not empty it is enough to fix a sixth point in $\mathbb{P}^3$ and compute the Hilbert function of the ideal $J = \bigcap_{i=0}^{5} \mathfrak{p}_i^2$, which we have already done in Section 2.4 with respect to the point $\mathfrak{p}_5 = (2x_0 - x_1, 3x_0 - x_2, 4x_0 - x_3)$. The projective space bundle B_1 over U associated to E is a rational variety embedded in B_0 as a dense open subset, and mapping dominantly onto $\mathcal{Q}$, which is therefore at least unirational.

Since $\mathcal{Q}$ is irreducible, because we have seen in the theorem that it is an open subset of an irreducible variety, to complete the proof it is now enough to compute the differential of the projection at one general point and show that the dimension of $\mathcal{Q}$ is 13. To this purpose the following procedure was applied to the standard test example of Section 2.4 (see Section 4.2 for details). Work over an affine subset of the product, where B_0 is given by four equations $F_1, \dots, F_4$ inside the polynomial ring $\mathbb{K}[t_1, \dots, t_{17}]$, and realise the tangent space of B_0 at the point P as the linear space in $\mathbb{A}^{17}$ corresponding to the ideal generated by the Jacobian equations $\sum_j \partial F_i / \partial x_j|_P t_j$. Then find the intersection of this ideal with the subalgebra $\mathbb{K}[t_1, \dots, t_{14}]$. □

Our goal is now to define a rational morphism from $\mathcal{Q}$ to the moduli space $\mathcal{R}_5$ of double covers of curves of genus five. Applying Lemma 3.4 it is enough to define a natural way of constructing families of curves from families of surfaces; natural in the precise sense of category theory, that is commuting with fibred products.

Proposition 3.7. *There exists a morphism of schemes $\varrho\colon \mathcal{Q} \to \mathcal{R}_5$, from $\mathcal{Q}$ to the moduli space of double covers of curves of genus five.*

Proof. Recall the general constructions described earlier in the thesis about how to associate to any nodal quartic hypersurface in $\mathbb{P}^3$ a nodal sextic plane curve with a double cover.

We now use a gluing argument, starting off assuming the base scheme to be the spectrum of a ring A. In this case we can proceed as we did before, associating to the scheme $\mathscr{X} = \operatorname{Proj} A[x_0, \dots, x_3]/(u_2x_3^2 + 2u_3x_3 + u_4)$ the plane curve over A defined by the equation $u_3^2 - u_2u_4$. This association is also natural, in that it commutes with pull-backs. Indeed for any homomorphism of rings $A \to A'$ the pull-back of the family of curves just defined is given by the projective spectrum of the graded ring

$$(A \otimes A')[x_0, x_1, x_2]/(u_3^2 - u_2 u_4)$$

which is the same graded ring one would obtain by first pulling back the family of surfaces and then applying the correspondence. □

We want to use the morphism described above to prove the unirationality of $\mathcal{R}_5$, in other words, in view of the other results, we want to prove that ϱ is dominant. To this purpose we now describe a first simplification of the problem. As we have seen in Section 3.1 the space $\mathcal{R}_5$ maps dominantly onto $\mathcal{M}_5$, the moduli space of curves of genus five, so we can consider the composition of morphisms to get the commutative diagram

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{\varrho} & \mathcal{R}_5 \\ & \searrow^{\theta} & \downarrow{\scriptstyle \eta} \\ & & \mathcal{M}_5 \end{array}$$

We now have the following result.

Lemma 3.8. *The morphism ϱ is dominant if and only if θ is dominant.*

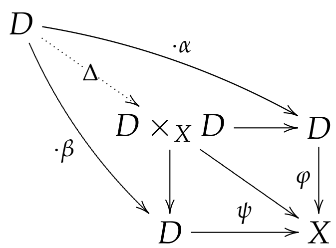
Proof. Given that η is a dominant morphism between irreducible spaces, if ϱ is also a dominant morphism the conclusion follows immediately. To see the converse it is probably more intuitive to proceed by contradiction. If ϱ was not dominant then the scheme theoretic image of $\mathcal{Q}$ inside $\mathcal{R}_5$ should be of lower dimension than $\dim \mathcal{R}_5 = 12$, because $\mathcal{R}_5$ is irreducible (see Section 3.1). But in this case the scheme theoretic image of the composition $\eta \circ \varrho$ would also be of lower dimension than 12. This is clearly not possible because assuming θ to be dominant is equivalent to assuming the dimension of its image to be 12, because $\mathcal{M}_5$ is also irreducible. □

To complete the argument we must now focus our attention to the morphism θ. In particular we look for sufficient conditions for it to be dominant, i.e. generically surjective.

Lemma 3.9. *Let $u\colon X \to X'$ be a morphism of algebraic schemes, with X' irreducible. Then the scheme theoretic image of u is the whole of X' if and only if there exists a point $P \in X$ such that the differential $\mathrm{d}u_P\colon T_{X,P} \to T_{X',u(P)}$ is surjective.*

Proof. Since X' is irreducible the scheme theoretic image of u will be the whole of X' if and only if the dimension of at least one of its irreducible components equals the dimension of X'. Now it is enough to recall that the dimension of the irreducible component of $\overline{u(X)}$ containing a regular point $u(P)$ is given by the rank of the differential. □

Following Harris and Morrison [26, §1.C] we can realise the tangent space to any scheme X at a closed point P as the set of maps from $D = \operatorname{Spec} \mathbb{K}[\varepsilon]/(\varepsilon^2)$ to X centred at P (that is, mapping the unique closed point $\mathbf{0} \in D$ to P). The linear operations on vectors are then given by means of fibred products, in that for any $\alpha, \beta \in \mathbb{K}$ the vector $\alpha\varphi + \beta\psi$ will be defined as



For any morphism $u\colon X \to X'$ and any closed regular point $P \in X$ the differential $\mathrm{d}u_P\colon T_{X,P} \to T_{X',u(P)}$ will now be simply given by the composition of morphisms $\varphi \mapsto u \circ \varphi$.

Lemma 3.10. *Let $u\colon X \to X'$ be a morphism of schemes, and let $P \in X$ be a closed point. Then the differential of $\mathrm{d}u$ of u is given by the restriction of the natural transformation $h(u)\colon h_X \to h_{X'}$ over D to the tangent space of X in the sense above.*

Proof. To see this we just need to unravel the definitions. In accordance with the previous discussion the tangent space of X at P is in the first place a subset of $\operatorname{Mor}(D, X)$, and the differential operates on this subset precisely as does $h(u)_D\colon \operatorname{Mor}(D, X) \to \operatorname{Mor}(D, X')$. □

3.3 Families of Curves and their Dimension

Let C be a canonically embedded curve of genus five, which we assume to be given by the complete intersection of three quadrics in $\mathbb{P}^4$. According to *Petri's Theorem* (see Arbarello, Cornalba, Griffiths and Harris [1], §III.3) we are thus assuming C to be canonically embedded and not trigonal.

Theorem 3.11 (Petri)**.** *Let C be a canonical curve of genus 5. Then the ideal of C is generated by three quadrics, unless C is trigonal in which case it is generated by three quadrics and three cubics.*

Given a family $\mathcal{F}$ of deformations of C, we want to know its dimension in terms of non-isomorphic curves, more precisely we look for a sufficient condition for its base to be generically surjective onto the moduli space $\mathcal{M}_5$ of genus five curves. To do this we can follow Sernesi [49] and realise $\mathcal{F}$ as a fibred product diagram

$$\mathcal{F}:\quad \begin{array}{ccc} C & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}\mathbb{K} & \xrightarrow{s} & S \end{array}$$

where f is flat and surjective, and S is the scheme of parameters, that is the base of the family. The question will then reduce to the computation of the *Kodaira-Spencer* map of $\mathcal{F}$, in that all the possible deformations of C are parametrised by the cohomology group $H^1(C, T_C)$ and there exists a linear map $\kappa_{\mathcal{F}} \colon T_{S,s} \to H^1(C, T_C)$ whose image consists precisely of those deformations contained in $\mathcal{F}$. Here $H^1(C, T_C)$ should be thought of as the tangent space to $\mathcal{M}_5$ at the point corresponding to C.

Observe that $H^1(C, T_C)$ is isomorphic to the vector space $H^0(C, K_C^{\otimes 2})^*$, indeed the tangent sheaf T_C is dual to the canonical sheaf K_C and by Serre duality we have

$$H^1(C, \mathscr{F}) = H^0(C, \mathscr{F}^\vee \otimes K_C)^*$$

for any invertible sheaf $\mathscr{F}$. Since we are assuming the curve to be canonically embedded the sheaf $K_C^{\otimes 2}$ will be in this case just the twisted sheaf $\mathscr{O}_C(2)$, thus if I is the ideal inside $R = \mathbb{K}[x_0, \dots, x_4]$ defining the curve C the cohomology group above will be given by the quotient R_2/I_2.

We need to investigate necessary and sufficient conditions for two such curves to be isomorphic. First we observe that two canonically embedded curves are isomorphic if and only if they are projectively equivalent. Thus we are now reduced to deciding how many projectively equivalent curves there are among the fibres of the morphism f.

Lemma 3.12. *Two canonically embedded curves of genus g are isomorphic if and only if they are projectively equivalent.*

Proof. A consequence of Riemann-Roch is that the only linear system g^{g-1}_{2g-2} of degree $2g-2$ and dimension $g-1$ on a curve of genus g is the canonical series [Arbarello et al., 1, p.10]. Therefore if $\varphi\colon C_1 \to C_2$ is an isomorphism, we can conclude that the pull-back of the canonical series of C_2 is the canonical series of C_1. But then we can define an automorphism of $\mathbb{P}^{g-1}$ inducing φ, concluding that C_1 and C_2 are projectively equivalent. □

In terms of projective equivalence we can realise a model containing all the possible canonical curves of genus five by considering the Grassmannian of dimension three subspaces of the vector space R_2. Projective equivalence is then given by the action of the group SL(5) and we are looking at the space of the orbits.

But the Grassmannian itself can be realised as an orbit space, this time under the action of GL(3), as follows. Let V be the open set inside the 45-dimensional vector space $R_2 \times R_2 \times R_2$ given by all the possible bases of dimension three subspaces in R_2, and consider the action of GL(3) whose orbits are all the possible bases for a given subspace. This is the obvious action

$$M({}^t\underline{x}Q_1\underline{x}, {}^t\underline{x}Q_2\underline{x}, {}^t\underline{x}Q_3\underline{x}) = \left(\sum_{i=1}^{3} m_{1i}{}^t\underline{x}Q_i\underline{x}, \sum_{i=1}^{3} m_{2i}{}^t\underline{x}Q_i\underline{x}, \sum_{i=1}^{3} m_{3i}{}^t\underline{x}Q_i\underline{x}\right),$$

where Q_i is a symmetric matrix and ${}^t\underline{x}$ is the row vector $(x_0, \ldots, x_4)$. We now consider the action of SL(5) whose orbits are all the possible projectively equivalent bases, which is the following

$$N({}^t\underline{x}Q_1\underline{x}, {}^t\underline{x}Q_2\underline{x}, {}^t\underline{x}Q_3\underline{x}) = ({}^t\underline{x}\,{}^tNQ_1N\underline{x}, {}^t\underline{x}\,{}^tNQ_2N\underline{x}, {}^t\underline{x}\,{}^tNQ_3N\underline{x})$$

It is important to observe that the two actions commute, thus they don't interfere with each other and we can regard one as acting on the orbit space of the other.

In order to investigate properties of a family of deformations of the curve C around its central fibre it is enough to consider the case in which the base scheme is affine and given by the spectrum of the ring $A = \mathbb{K}[t_0, \dots, t_N]/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal generated by $t_0, \dots, t_N$ which is assumed to correspond (as a point of Spec A) to the curve C. In this situation the scheme $\mathcal{C}$ will be defined by the projective spectrum of $A[x_0, \dots, x_4]/(F_1, F_2, F_3)$, where the coefficients of F_i depend linearly on the parameters t_i. In other words we can write every polynomial F_i as a sum

$$F_i = H_i + \sum_{j=0}^{N} t_j H_{ij},$$

where H_i and H_{ij} are quadric polynomials inside $\mathbb{K}[x_0, \dots, x_4]$. The N triplets of quadric polynomials (H_{1j}, H_{2j}, H_{3j}) generate the linear space inside $\mathbb{A}^{45}$ tangent to the family at the central point $s = (H_1, H_2, H_3)$. The problem is to determine whether or not this linear space fills the tangent space to $\mathcal{M}_5$; the strategy we are going to adopt is to work inside the tangent space to V at s, constructing a basis for all the trivial deformations and then check how many of the above triplets lie inside this linear space; keeping in mind that the trivial deformations are those given by the group actions described above.

Around any point $v = ({}^t\underline{x}Q_1\underline{x}, {}^t\underline{x}Q_2\underline{x}, {}^t\underline{x}Q_3\underline{x})$ in V the action of the two groups is linearised by the action of the corresponding Lie algebras, so a system of generators for the linear space tangent to the orbit passing through v is simply determined by applying a basis for the Lie algebra to it. Now the Lie algebra $\mathfrak{gl}(3)$ is simply the whole space of three-by-three matrices and its action is the same as the one of GL(3), thus we obtain a first set of trivial deformations given by the nine vectors

$$(H_1, 0, 0), (H_2, 0, 0), \dots, (0, 0, H_2), (0, 0, H_3)$$

The algebra $\mathfrak{sl}(5)$ instead is given by the vector space of traceless 5×5 matrices [see Fulton and Harris, 20, §8.2] and its action is determined as follows

$$\begin{aligned} {}^t(N\underline{x})Q_i(N\underline{x}) &= {}^t\underline{x}\,{}^t(I+\varepsilon\Delta)Q_i(I+\varepsilon\Delta)\underline{x} \\ &= {}^t\underline{x}(Q_i+\varepsilon({}^t\Delta Q_i + Q_i\Delta))\underline{x} \\ &= {}^t\underline{x}Q_i\underline{x} + \varepsilon\underline{x}({}^t\Delta Q_i + Q_i\Delta)\underline{x} \end{aligned}$$

thus letting Δ vary among a basis for $\mathfrak{sl}(5)$ we get another set of trivial deformations given by the 24 vectors

$$({}^t\Delta Q_1 + Q_1\Delta, {}^t\Delta Q_2 + Q_2\Delta, {}^t\Delta Q_3 + Q_3\Delta).$$

Here one should first write the associated symmetric matrix Q_i to each of the polynomials H_i; to keep this in mind we can write each of the entries above rather improperly as ${}^t\Delta H_i + H_i\Delta$.

Construct now the matrix $M_{\mathcal{F}}$ as follows.

$$M_{\mathcal{F}} : \begin{pmatrix} H_{11} & H_{21} & H_{31} \\ H_{12} & H_{22} & H_{32} \\ \vdots & \vdots & \vdots \\ H_{1N} & H_{2N} & H_{3N} \\ H_1 & 0 & 0 \\ H_2 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & H_2 \\ 0 & 0 & H_3 \\ D_{21}H_1 + H_1D_{12} & D_{21}H_2 + H_2D_{12} & D_{21}H_3 + H_3D_{12} \\ D_{31}H_1 + H_1D_{13} & D_{31}H_2 + H_2D_{13} & D_{31}H_3 + H_3D_{13} \\ \vdots & \vdots & \vdots \\ D_{55}H_1 + H_1D_{55} & D_{55}H_2 + H_2D_{55} & D_{55}H_3 + H_3D_{55} \end{pmatrix}$$

The first N rows are given by the family $\mathcal{F}$, they are tangent vectors at the central point $s = (H_1, H_2, H_3)$. The second set of 9 rows is given by the tangent vectors to the orbit of the GL(3)-action. The last 24 rows are the tangent vectors to the orbits of the SL(5)-action described above. We have considered here the basis for the vector space of traceless 5×5 matrices given by the matrix D_{ij} with a unique nonzero entry (which will be a 1) in position $_{ij}$ for $i \neq j$ and

$$
\begin{aligned}
D_{22} &= \operatorname{diag}(1, -1, 0, 0, 0) \\
D_{33} &= \operatorname{diag}(1, 0, -1, 0, 0) \\
D_{44} &= \operatorname{diag}(1, 0, 0, -1, 0) \\
D_{55} &= \operatorname{diag}(1, 0, 0, 0, -1)
\end{aligned}
$$

The rows of the matrix $M_{\mathcal{F}}$ generate a linear space, which is the span inside the tangent space to V of the three linear spaces tangent respectively to the given family and to each of the two orbits through s. To determine the dimension of this span we now need to compute the rank of $M_{\mathcal{F}}$.

Proposition 3.13. *Let C be the complete intersection of three quadrics in $\mathbb{P}^4$, and let $\mathcal{F}$ be a family of deformations of C as above. Assume also that the dimension of $\mathcal{F}$ is at least 12, in other words $N \geq 12$. If the rank of the matrix $M_{\mathcal{F}}$ is maximal then the Kodaira-Spencer map of $\mathcal{F}$ is surjective.*

Proof. First observe that in the matrix $M_{\mathcal{F}}$ there are 45 columns, and under our assumptions there are at least 45 rows. When the rank of the matrix $M_{\mathcal{F}}$ is maximal the span of the three vector spaces, the two corresponding to trivial deformations and the one given by the family, is the whole of the tangent space to V at the point s. Thus we are guaranteed the existence of enough linearly independent deformations, namely 12, to fill the tangent space to $\mathcal{M}_5$. □

Looking back now at the morphism $\varrho \colon \mathcal{Q} \to \mathcal{R}_5$ defined in Proposition 3.7 we put everything together and show how the matrix $M_{\mathcal{F}}$ described above can help us proving the unirationality of $\mathcal{R}_5$. We have already observed in Lemma 3.8 that it is enough to prove that the morphism $\theta \colon \mathcal{Q} \to \mathcal{M}_5$ is generically surjective, and in Lemma 3.9 we have reduced the argument to showing that the differential of θ at one point is surjective. By Lemma 3.4 the morphism θ corresponds to a natural transformation

$$
h(\theta) \colon h_{\mathcal{Q}} \longrightarrow h_{\mathcal{M}_5}
$$

that we actually have defined in Proposition 3.7.

Theorem 3.14. *The moduli space $\mathcal{R}_5$ of étale two-sheeted coverings of curves of genus five is unirational.*

Proof. We fix a regular point $\mathbb{X}$ inside $\mathcal{Q}$, and compute the differential of θ at $\mathbb{X}$ as in Lemma 3.10 by restricting the natural transformation over D

$$h(\theta)_D \colon \mathbf{Q}(D) \longrightarrow \mathbf{m}_5(D)$$

To this purpose we start with a particular family of deformations of $\mathbb{X}$, the one we have defined in Proposition 3.5. Recall the definition of $B_0 \subseteq \mathbb{P}^{14} \times \mathbb{P}^3$ and consider the family $\mathcal{F}$ given by the following fibred product diagram

$$\mathcal{F}\colon \quad \begin{array}{ccc} \mathbb{X} & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow f \\ \operatorname{Spec}\mathbb{K} & \xrightarrow{b} & B_0 \end{array}$$

where b is the point of B_0 corresponding to $\mathbb{X}$. Pulling back $\mathcal{F}$ along a basis of vectors $\varphi_i \colon D \to B_0$ ($i = 1, \dots, 13$) for the tangent space of B_0 at b we obtain linearly independent elements of the tangent space $T_{\mathcal{Q},\mathbb{X}} \subset \mathbf{Q}(D)$, which by the way will form a basis for $T_{\mathcal{Q},\mathbb{X}}$ because the dimension coincides.

Applying the natural transformation $h(\theta)_D$ to any of these we construct 13 first order deformations of the canonically embedded curve $\widetilde{C_{\mathbb{X}}}$ as in Section 2.3, which will give rise to as many rows of the matrix $M_{\mathcal{F}}$ described above.

It is now enough to check if the rank of this matrix is maximal for a particular example, which may as well be the test curve described in Section 2.4. Indeed we performed the required computations on the test surface $\mathbb{X}$, and obtained a positive answer: the rank is maximal. In Section 4.4 we describe the details and explain the computer code used with *Macaulay 2*. □

In the 1980s Donagi [12] proved the unirationality of $\mathcal{R}_6$ at about the same time as Catanese [5] proved that $\mathcal{R}_4$ is in fact rational, but since then there has been a gap between four and six, which we have now filled with the above theorem. Besides, as we have seen at the end of Chapter 1 we also obtain as a consequence a new proof of a theorem of Clemens [8]

Corollary 3.15. $\mathcal{A}_4$ *is unirational.*

This result follows in fact from the previous theorem because the Prym map $p_5 \colon \mathcal{R}_5 \to \mathcal{A}_4$ is generically surjective [see Beauville, 4]. The original proof of Clemens differs from the present in that it relied upon intermediate Jacobians.

Chapter 4

Computations

The last part of this thesis is devoted to the computations. Having already made clear in Section 2.4 the reasons why the results obtained here are relevant, in what follows we just describe how we were able to construct an explicit example, and to test it for all the properties needed. The computer algebra package which the code refers to is *Macaulay 2* [21], so in principle one should be able to copy it as it stands and obtain at least similar results; if the online documentation doesn't help there is also the book edited by Eisenbud, Grayson and Stillman [16]. When describing the code, input data will be preceded by `i:` while output data will be preceded by `o:`, for no particular reason at all the ground field is always $\mathbb{F}_{101}$.

4.1 The Standard Test Example

We fix the coordinate ring of $\mathbb{P}^3$ and the six points $P_0, \ldots, P_5$, which will be the double points of the surface.

```
i: R = ZZ/101[x_0..x_3];
i: P0 = ideal(x_0,x_1,x_2);
i: P1 = ideal(x_0,x_1,x_3);
i: P2 = ideal(x_0,x_2,x_3);
i: P3 = ideal(x_1,x_2,x_3);
i: P4 = ideal(x_0-x_1,x_0-x_2,x_0-x_3);
i: P5 = ideal(x_1-2*x_0,x_2-3*x_0,x_3-4*x_0);
```

The surface $\mathbb{X}$ will be defined by a single irreducible homogeneous polynomial F of degree four contained in the intersection $I = \bigcap_{i=0}^{5} \mathfrak{p}_i^2$, and as such it can be randomly selected. More precisely we ask *Macaulay 2* to compute a system of generators for the ideal I, which will be returned in the form of a matrix as follows.

```
i: I = intersect(P0^2,P1^2,P2^2,P3^2,P4^2,P5^2);
i: polys = gens I;
              1       11
o: Matrix R  <--- R
```

Then we instruct the program to compute a random matrix of homogeneous polynomials in R such that when multiplied with the above the result is a form of degree four.

```
i: MM1 = random(source polys, R^{-4})
              11       1
o: Matrix R   <--- R
i: F1mat = polys * MM1;
              1       1
o: Matrix R  <--- R
```

Observe that `F1mat` is in fact a one-by-one matrix, so that the equation we are after will be the determinant `F1 = det F1mat`. The result must now be tested, firstly to ensure its singular locus amounts to just the six double points and then to ensure it verifies the generic condition described in Proposition 2.4.

```
i: D1 = diff F1mat;
i: F1sing = ideal(D1);
i: hilbertPolynomial F1sing
o = 6*P
       0
o: ProjectiveHilbertPolynomial
```

The Hilbert polynomial ensures us that the singular locus consists of just the six points we fixed. Next we turn to the condition on the cones over P_0. The equation `F1` will be of the form $u_2 x_3^2 + 2u_3 x_3 + u_4$ and we want to compute the forms u_i first. The command `coefficients` will do, but we must remember that it returns a list of two lists, one containing the powers of x_3 present in the expression and the other the actual coefficients.

```
i: CoeffListF1 = coefficients({3},F1mat);
i: CoeffF1 = CoeffListF1_{1};
i: F1cones = ideal(CoeffF1);
i: hilbertPolynomial F1cones
o = 24*P
        0
o: ProjectiveHilbertPolynomial
```

Again the Hilbert polynomial assures us that the condition is satisfied, because saying that this intersection consists of only a finite number of points is equivalent to saying it is given by P_0 with some multiplicity. Indeed u_i is a cone over P_0 for each i, so the intersection contains P_0 only or otherwise it contains a line. Now we can compute the equation of the plane curve C_X as follows.

```
i: F1curves = gens F1cones;
i: G4mat = F1curves*matrix{{1},{0},{0}};
i: G4 = det G4mat
i: G3mat = F1curves*matrix{{0},{1},{0}}
i: G3 = 1/2*(det G3mat)
i: G2mat = F1curves*matrix{{0},{0},{1}}
i: G2 = det G2mat
i: F1 = det F1mat
i: F1 == G2*x_3^2+2*G3*x_3+G4
o = true
i: CX = G3^2-G2*G4;
```

In translating the instructions given to *Macaulay 2* into our notation the polynomials `G2`, `G3` and `G4` will become u_2, u_3 and u_4. Thus the three homogeneous forms above, which passed all the tests in *Macaulay 2*, were the following

$$\begin{aligned}
u_2 =&19x_0^2 - 33x_0x_1 + 50x_1^2 - 13x_0x_2 + 50x_1x_2 - 15x_2^2 \\
u_3 =&- 2x_0^2x_1 - 35x_0x_1^2 - 18x_0^2x_2 - 8x_0x_1x_2 - 36x_1^2x_2 - 4x_0x_2^2 + 45x_1x_2^2 \\
u_4 =&- 38x_0^2x_1^2 - 32x_0^2x_1x_2 - 32x_0x_1^2x_2 - 6x_0^2x_2^2 - 38x_0x_1x_2^2 + 2x_1^2x_2^2
\end{aligned}$$

Now we are left with a plane curve, which we know has five ordinary nodes, and we want to compute its normalisation. In order to do so we must blow up the plane in five points, and for this we follow Griffiths and Harris [22, §4.4] and consider the linear system of cubic curves passing through the points.

```
i: R = ZZ/101[x_0,x_1,x_2]
i: P1 = ideal(x_0,x_1);
i: P2 = ideal(x_0,x_2);
i: P3 = ideal(x_1,x_2);
i: P4 = ideal(x_0-x_1,x_0-x_2);
i: P5 = ideal(x_1-2*x_0,x_2-3*x_0);
i: canC = basis(3,intersect(P1,P2,P3,P4,P5));
i: canC = canC ** R;
i: canC = super canC;
           1       5
o: Matrix R  <--- R
```

Quoting from the online documentation of *Macaulay 2*: "Some explanation regarding the `basis` command is needed here. `canC` is a matrix whose target is the ideal of the intersection of these five points, and whose source is a free module over the coefficient ring. For our purposes, there are two problems with this. The first is that we want a map where both the source and target have the base ring R. This can be accomplished by tensoring with R. The second problem is that the image of a basis element is not obviously in the ideal: it is represented in terms of the generators of I. This can be alleviated by applying `super`: this takes a homomorphism $f\colon M \to N$, where N is a submodule of a quotient module F/I, and returns the homomorphism $f\colon M \to F/I$." Now we construct the embedding of the curve `CX` in $\mathbb{P}^4$ as follows.

```
i: S = (coefficientRing R)[y_0..y_4]
i: TC = R/ideal CX
i: ff = map(TC,S,substitute(canC, TC));
i: cancurve = mingens ker ff;
           1       3
o: Matrix S  <--- S
```

The kernel of ff is the ideal of the curve $\widetilde{C_{\mathbb{X}}}$, which we expect from Petri's Theorem (3.11) to be generated by three quadrics. Indeed the three generators we have found are quadrics.

```
i: Q1 = cancurve * matrix{{1},{0},{0}};
i: det Q1
                2
o = y y  + y  - y y  - 34y y  + 33y y
     1 2    2    0 3      2 3      2 4
i: Q2 = cancurve * matrix{{0},{1},{0}};
i: det Q2
                 2
o = y y  - 2y  - 33y y  - y y  + 35y y
     0 2     2       2 3   1 4      2 4
i: Q3 = cancurve * matrix{{0},{0},{1}};
i: det Q3
     2              2                     2
o = y  - 48y y  + 5y  + ... + 17y y  - 19y
     0       0 1     1            3 4      4
```

In order to check that the double cover of $\widetilde{C_{\mathbb{X}}}$ constructed in Proposition 2.8 is not trivial we must show that the surface Σ is irreducible. This amounts to proving that the following ideal in $\mathbb{K}[x,y][\xi_0,\xi_1,\xi_2]$, homogeneous in ξ_0,ξ_1,ξ_2,

$$(-\xi_2^2+\xi_1^2G_2+2\xi_1\xi_0G_3+\xi_0^2G_4, G_3^2-G_2G_4)$$

is prime. But in fact we are working here over the open subset $D_h(x_i)$ of $\mathbb{P}^2$, and G_d is the polynomial obtained from u_d by multiplication by $1/x_i^d$, so that actually we are to check that three (very similar) ideals are prime. First we compute the equations of u_d (d=2,3,4) over the open affine subset $D_h(x_i)$ by instructing for instance `G20 = substitute(G2,{x_0=>1})`, and similarly for `G30` and `G40` with the obvious association of u_2 over $D_h(x_0)$ with `G20`. Then we define the ideal to test

```
i: R = ZZ/101[x_1,x_2,z_0..z_2];
i: G20 = 50*x_1^2+50*x_1*x_2-15*x_2^2-33*x_1-13*x_2+19
i: G30 = -36*x_1^2*x_2+ ... -18*x_2
```

```
i: G40 = 2*x_1^2*x_2^2- ... -6*x_2^2
i: I = ideal(G30^2+G20*G40,
            -z_2^2+z_1^2*G20+2*z_1*z_0*G30+z_0^2*G40);
```

However we are not going to check primality directly, but instead we work inside the 5-dimensional affine space and cut the threefold with a codimension two linear space H. This is enough because if the threefold was reducible, say given by the union $Z_1 \cup Z_2$, when cutting with H one would see the union of $(Z_1 \cap H)$ and $(Z_2 \cap H)$. So if what we get is irreducible one of those must be empty. But if $Z_2 \cap H$ is empty, Z_2 must be.

```
i4 : I = ideal(G30^2+G20*G40,
              -z_2^2+z_1^2*G20+2*z_1*z_0*G30+z_0^2*G40,
              z_1-1,z_2-1);
i5 : decompose I
o6 : List
i7 : toString o6
o7 = {ideal(z_2-1, z_1-1,
      x_1^4*x_2^2- ... +22*x_2^2,
      x_1^2*x_2^2*z_0^2- ... +9,
      x_1^4*x_2*z_0^2+ ... -19,
      -6*x_1^5*x_2*z_0+ ... +25,
      x_1*x_2^5*z_0^2+ .... -37,
      x_1^3*x_2^4*z_0- .... +29,
      x_2^6*z_0^3+ ... -50)}
```

The function `decompose` returns the list of the irreducible components of an ideal, thus the output is showing that `I` is irreducible. Clearly we are documenting here only one of three necessary computations, but the computations on the other two affine pieces are identical.

4.2 The dimension of the family of surfaces

In this section we deal with the family of quartic hypersurfaces we described in Proposition 3.5. We want to compute the differential of the projection from

$\mathbb{P}(I_4) \times \mathbb{P}^3$ to $\mathbb{P}(I_4)$, restricted to the subvariety B_0 of the product given by the equations

$$\left\{(F,\mathfrak{p}) \mid F \in \mathfrak{p}^2\right\} = \left\{(F,P) \;\middle|\; F(P) = \frac{\partial F}{\partial x_i}(P) = 0\right\}$$

We first need to compute a basis for the degree four part of the ideal, and this is achieved as follows.

```
i: R = ZZ/101[x_0..x_3];
i: P0 = ideal(x_0,x_1,x_2);
i: ...
i: P4 = ideal(x_0-x_1,x_0-x_2,x_0-x_3);
i: Surfs = basis(4,intersect(P0^2,P1^2,P2^2,P3^2,P4^2));
i: Surfs = Surfs ** R;
i: Surfs = super Surfs;
          1       15
o: Matrix R  <--- R
```

And we also have to write our central fibre over this basis, which is given by typing simply `F1 // Surfs`.

```
o: matrix {{-33}, {17}, {19}, {-16}, {13}, {-36}, {-4}, {2},
           {-38}, {-46}, {-6}, {-32}, {-1}, {-32}, {-38}};
```

Now we wish to work over an affine open subset, which means that we shall have fourteen coordinates coming from the above basis and three coming from projective space. The following code produces the equations we are interested in over the relevant open set.

```
i: A = ZZ/101[t_1..t_14];
i: R = A[x_0..x_3];
i: MatSurf = Surfs * matrix{{-33},{t_1}..{t_14}};
i: MatSurf = substitute(MatSurf,{x_0=>1})
```

Note that we are choosing an open subset where $x_0 \neq 0$, therefore we now pass to a polynomila ring in three variables x_1, x_2, x_3. Besides we construct a matrix containing all the equations we need, four in total.

```
i: A = ZZ/101[t_1..t_14];
```

```
i: R = A[x_1..x_3];
i: MatSurf = as above
i: MatDiff = diff MatSurf;
i: MatFam = MatSurf|MatDiff;
```

Now we operate over an open affine subset, and again we need to change the ideal for the number of variables to be correct. We compute this time the Jacobian matrix of the four equations, or in other words the tangent space of B_0 (embedded in $\mathbb{A}^{17}$) at the point corresponding to our surface $\mathbb{X}$.

```
i: A = ZZ/101[t_1..t_14,x_1..x_3];
i: MatFam = as above
i: FamDiff = jacobian MatFam
i: FamJac = substitute(FamDiff,{t_1=>17, t_2=>19,
       t_3=>-16, t_4=>13, t_5=>-36, t_6=>-4, t_7=>2,
       t_8=>-38, t_9=>-46, t_10=>-6, t_11=>-32, t_12=>-1,
       t_13=>-32, t_14=>-38, x_1=>2, x_2=>3, x_3=>4})
```

Finally we compute the intersection of the ideal of definition of the tangent space with the subalgebra given by the first fourteen variables.

```
i: R = ZZ/101[t_1..t_17];
i: FamJac = as above
i: TgSpId = FamJac * matrix{{t_1}..{t_17}};
          4       1
o: Matrix R  <--- R
i: TSI = ideal TgSpId;
i: MatPrj = matrix{{t_1..t_14}};
i: S = (coefficientRing R)[y_1..y_14];
i: TC = R/TSI;
i: ff = map(TC,S,substitute(MatPrj, TC));
o: RingMap TC <--- S
i: mingens ker ff
          1       1
o: Matrix S  <--- S
```

4.3 The Family of Plane Curves

Starting from the family of quartic hypersurfaces we described in Proposition 3.5 we compute here the infinitesimal deformations of the standard test example $\mathbb{X}$ and the associated family of deformations of sextic plane curves. We wish to work on an open affine subset of B_0 centred at $\mathbb{X}$, so we first fix coordinates `a,b,c,t_1..t_14`. We use the basis `surfs` for quartic hypersurfaces with nodes at five fixed points that we computed above.

```
i: A = ZZ/101[a,b,c,t_1..t_14];
i: R = A[x_0..x_3];
i: Surfs = as above
i: MatSurf = matrix {{-33}, {17+t_1}, {19+t_2},
     {-16+t_3}, {13+t_4}, {-36+t_5}, {-4+t_6},
     {2+t_7}, {-38+t_8}, {-46+t_9}, {-6+t_10},
     {-32+t_11}, {-1+t_12}, {-32+t_13}, {-38+t_14}}
i: Surface = det(Surfs*MatSurf);
i: DffSurf = diff Surface;
i: Eqtn1 = substitute(Surface,
           {x_0=>1, x_1=>2+a, x_2=>3+b, x_3=>4+c});
i: Eqtn2 = substitute(DffSurf,
           {x_0=>1, x_1=>2+a, x_2=>3+b, x_3=>4+c});
```

In order to define coordinates over B_0 we need to solve the linear system in `t_1..t_14` with respect to `a,b,c`; we can do it by computing a Gröbner basis for the ideal of the equations (function gb).

```
i1 : A = ZZ/101[a,b,c];
i2 : K = frac A;
i3 : R = K[t_1..t_14];
i4 : Eqtn1 = as above
i5 : Eqtn2 = as above
i6 : EqSys = ideal(Eqtn1)+ideal(Eqtn2);
i7 : Eqns = generators gb EqSys;
              1       4
o7 : Matrix R   <--- R
```

At this point some adjustments are needed. The solution of the system allows us to replace four variables with polynomials, and we compute these four. In this case the variables happen to be `t_5,t_3,t_2,t_1` so we name the polynomials `G5,G3,G2,G1`. After this operation we will be left with an asymmetric set of variables, so we rename them.

```
i3 : R = K[t_1..t_14,s_1..s_10];
i4 : G1 = as above
i5 : H1 = substitute(G1, {t_4=>s_1, t_6=>s_2,
     t_7=>s_3, t_8=>s_4, t_9=>s_5, t_10=>s_6,
     t_11=>s_7, t_12=>s_8, t_13=>s_9, t_14=>s_10})
```

Now we substitute the solution of the linear system inside the expression of the family.

```
i1 : A = ZZ/101[a,b,c];
i2 : K = frac A;
i3 : B = K[s_1..s_10];
i4 : R = B[x_0..x_3];
i5 : H1 = as above
i6 : H2 = as above
i7 : H3 = as above
i8 : H5 = as above
i9 : Surfs =  as above
i10 : MatSurf = matrix {{-33}, {17+H1}, {19+H2},
         {-16+H3}, {13+s_1}, {-36+H5}, {-4+s_2},
         {2+s_3}, {-38+s_4}, {-46+s_5}, {-6+s_6},
         {-32+s_7}, {-1+s_8}, {-32+s_9}, {-38+s_10}};
i11 : Surface = det(Surfs*MatSurf);
```

The coefficients of the quartic polynomial `surface` are rational functions on `a,b,c`. In order to deal with an infinitesimal family we then clear denominators.

```
i1 : A = ZZ/101[a,b,c];
i2 : K = frac A;
i3 : B = K[s_1..s_10];
i4 : R = B[x_0..x_3];
```

```
i5 : Surface = as above
i6 : Surface = (a^3*b*c- ... -12*a+4*b-5*c-4)*Surface
i7 : Surface = (b^2+5*b+6)*Surface
i8 : Surface = (c+4)*Surface
i9 : Surface = (a-b-1)*Surface
i10 : Surface = 38/12*Surface
```

Finally we can read the equation infinitesimally around the point $\mathbb{X}$, just by taking the quotient for the maximal ideal squared.

```
i1 : A = ZZ/101[a,b,c,s_1..s_10];
i2 : JJ = ideal(a,b,c,s_1..s_10);
i3 : B = A/JJ^2
i4 : R = B[x_0..x_3];
i5 : Surface = as above
```

We are now ready to compute the equation for the family of curves, but again *Macaulay 2* doesn't allow us to work directly over the infinitesimal ring. So first we make the computation over the fraction field and then we clear the terms of higher order with a quotient.

```
i1 : A = ZZ/101[a,b,c,s_1..s_10];
i2 : K = frac A;
i3 : R = K[x_0..x_3];
i4 : surface = as above
i5 : CoeffListF1 = coefficients({3},surface)
i6 : CoeffF1 = CoeffListF1_{1}
i7 : F1cones = ideal(CoeffF1)
i8 : F1curves = gens F1cones
i9 : G4mat = F1curves*matrix{{1},{0},{0}}
i10 : G4 = det G4mat
i12 : G3mat = F1curves*matrix{{0},{1},{0}}
i13 : G3 = 1/2*(det G3mat)
i15 : G2mat = F1curves*matrix{{0},{0},{1}}
i16 : G2 = det G2mat
i18 : surface == G2*x_3^2+2*G3*x_3+G4
o18 = true
```

```
i19 : curves = G3^2-G2*G4
```

In what follows all *Macaulay 2* does is read the polynomial we input, which we take from above, and then write it back leaving out all the terms of higher order.

```
i1 : A = ZZ/101[a,b,c,s_1..s_10];
i2 : JJ = ideal(a,b,c,s_1..s_10);
i3 : B = A/JJ^2
i4 : R = B[x_0..x_2];
i5 : curves =
```

4.4 The Rank of a big Matrix

With reference to Section 3.3, and in particular to Theorem 3.14, we want to compute the matrix $M_{\mathcal{F}}$ where $\mathcal{F}$ is the family of quartic surfaces we described in Section 4.2 or more precisely for our current purposes the family `curves` of sextic plane curves defined in the previous section. We start by computing the part of $M_{\mathcal{F}}$ which doesn't depend on the family, but only on the central fibre. So, looking at the action of $\mathfrak{gl}(3)$ we need to write the coefficients of the three quadratic forms already encountered in Section 4.1.

```
i: R = ZZ/101[y_0..y_4];
i: F1 = y_1*y_2+y_2^2-y_0*y_3-34*y_2*y_3+33*y_2*y_4;
i: F2 = y_0*y_2-2*y_2^2-33*y_2*y_3-y_1*y_4+35*y_2*y_4;
i: F3 = y_0^2-48*y_0*y_1+ ... +17*y_3*y_4-19*y_4^2;
i: MM = basis(2,R)
i7 : MM1 = transpose(F1//MM)
i9 : MM2 = transpose(F2//MM)
i10 : MM3 = transpose(F3//MM)
```

Given the vectors `MM1`, `MM2` and `MM3` one can write down the corresponding 9×45 matrix by hand, which is exactly what we did. Let it be `MM9`. As for the action of $\mathfrak{sl}(5)$ we start by fixing a basis for the vector space of traceless 5×5 matrices, denoting `D24` the matrix with a unique nonzero entry (which will be a 1) in position $_{24}$ and by `DD3` the diagonal matrix $\mathrm{diag}(1, 0, 0, -1, 0)$. Then denoting by Q2 the symmetric matrix associated to the polynomial F2 above we do the following.

```
i: R = QQ[y_0..y_4];
i: MM = basis(2,R)
i: X = matrix{{y_0..y_4}}
i: Y = transpose X
i: F01P1 = det (X*(Q1*D01+D10*Q1)*Y)
i: F01P2 = det (X*(Q2*D01+D10*Q2)*Y)
i: F01P3 = det (X*(Q3*D01+D10*Q3)*Y)
i: C01P1 = transpose (F01P1 // MM)
i: C01P2 = transpose (F01P2 // MM)
i: C01P3 = transpose (F01P3 // MM)
i: C01 = C01P1|C01P2|C01P3
i: F02P1 = det (X*(Q1*D02+D20*Q1)*Y)
i: F02P2 = det (X*(Q2*D02+D20*Q2)*Y)
i: F02P3 = det (X*(Q3*D02+D20*Q3)*Y)
i: C02P1 = transpose (F02P1 // MM)
i: C02P2 = transpose (F02P2 // MM)
i: C02P3 = transpose (F02P3 // MM)
i: C02 = C02P1|C02P2|C02P3
i: slfive = C01||C02
i: ...
i: F24P1 = det (X*(Q1*DD4+DD4*Q1)*Y)
i: F24P2 = det (X*(Q2*DD4+DD4*Q2)*Y)
i: F24P3 = det (X*(Q3*DD4+DD4*Q3)*Y)
i: C24P1 = transpose (F24P1 // MM)
i: C24P2 = transpose (F24P2 // MM)
i: C24P3 = transpose (F24P3 // MM)
i: C24 = C24P1|C24P2|C24P3
i: slfive = slfive||C24
i: MatFam = slfive||MM9
                33       45
o: Matrix R   <--- R
i: rank MatFam
```

```
o = 33
```

For each first order deformation of the family curves mentioned above, we now need to compute the normalisation. But first we need to extract each equation from the equation of the family.

```
i: A = ZZ/101[a,b,c,s_1..s_10];
i: R = A[x_0..x_2];
i: curves = as above
i: C01 = substitute(curves,{a=>0, b=>0, c=>0, s_2=>0,
            s_3=>0, s_4=>0, s_5=>0, s_6=>0, s_7=>0,
            s_8=>0, s_9=>0, s_10=>0})
i:  ...
i: C10 = substitute(curves,{a=>0, b=>0, c=>0, ..., s_9=>0})
i: C11 = substitute(curves,{a=>0, b=>0, s_1=>0, ..., s_10=>0})
i: C12 = substitute(curves,{a=>0, c=>0, s_1=>0, ..., s_10=>0})
i: C13 = substitute(curves,{b=>0, c=>0, s_1=>0, ..., s_10=>0})
```

In Section 4.1 we have already encountered the algorithm for the computation of the normalisation of a nodal plane curve of this type. Observe that it relies on a basis of plane cubics through the five points we want to blow up. It is clear then that the set of deformations above consists of two disjoint parts: those deformations with nodes at five fixed points, and those with a non-fixed node. The former is the biggest, containing eleven deformations out of thirteen, indeed from the geometry of the problem one sees immediately that it contains also the parameter c. Thus we distinguish two cases, and start from the most common one. We denote the parameter for each 1-dimensional family we consider by e, meaning successively e=c, e=s_1, $\cdots$, e=s_10

```
i: A = ZZ/101[e]/ideal e^2;
i: K = frac A
i: R = K[x_0,x_1,x_2];
i: C = this is the deformation we are considering
```

The following set of commands computes the pull-back of the family above, in other words it computes the corresponding deformation of canonical curves.

```
i: canC =  as in Section 4.1
```

```
i: S = (coefficientRing R)[y_0..y_4];
i: TC = R/ideal C;
i: ff = map(TC,S,substitute(canC, TC));
i: cancurve = mingens ker ff
i: F1 = det( cancurve * matrix{{1},{0},{0}} )
i: F2 = det( cancurve * matrix{{0},{1},{0}} )
i: F3 = det( cancurve * matrix{{0},{0},{1}} )
```

Now we need to normalise the expressions, but *Macaulay 2* doesn't allow us to do it over this ring. We clear denominators and check that the central fiber coincides with our chosen one.

```
i: A = ZZ/101[e];
i: K = frac A
i: R = K[y_0..y_4];
i: Q1 = y_1*y_2+y_2^2-y_0*y_3-34*y_2*y_3+33*y_2*y_4
i: Q2 = y_0*y_2-2*y_2^2-33*y_2*y_3-y_1*y_4+35*y_2*y_4
i: Q3 = y_0^2-48*y_0*y_1+ ... +17*y_3*y_4-19*y_4^2
i: T1 = F1-Q1
i: T2 = F2-Q2
i: T3 = F3-Q3
i: MM = basis(2,R)
i: RP1 = transpose (T1 // MM)
i: RP2 = transpose (T2 // MM)
i: RP3 = transpose (T3 // MM)
i: Row01 = RP1|RP2|RP3
```

Lastly we attach the row just computed to the matrix we already have.

```
i: MatFam = MatFam||Row01
```

In the case of the parameters `a,b` the difference is of course in how we compute the normalisation. So we leave the rest untouched and go back to that computation. The following set of commands computes the pull-back of the family above, in other words it computes the corresponding deformation of canonical curves.

```
i: P1 = ideal(x_0,x_1);
```

```
i: P2 = ideal(x_0,x_2);
i: P3 = ideal(x_1,x_2);
i: P4 = ideal(x_0-x_1,x_0-x_2);
i: P5 = ideal(x_1-(2+a)*x_0,x_2-3*x_0);
```

or

```
i: P5 = ideal(x_1-2*x_0,x_2-(3+b)*x_0);
i: canC = basis(3,intersect(P1,P2,P3,P4,P5));
i: canC = canC ** R;
i: canC = super canC;
i: C = this is the deformation we are considering
i: S = (coefficientRing R)[y_0..y_4];
i: TC = R/ideal C;
i: ff = map(TC,S,substitute(canC, TC))
i: cancurve = mingens ker ff
i: F1 = det( cancurve * matrix{{1},{0},{0}} )
i: F2 = det( cancurve * matrix{{0},{1},{0}} )
i: F3 = det( cancurve * matrix{{0},{0},{1}} )
```

After this lengthy set of computations we were left with a 46×45 matrix, whose entries are actually integers and whose rank we want to be maximal for our result to hold. The name we actually gave to this matrix for *Macaulay 2* to handle at this stage was an emblematic `hope`. We conclude this thesis with the answer we received, as it appeared on the screen one day, leaving the reader to imagine what happened next around that same screen. . .

```
                 46       45
o28 : Matrix R   <--- R
i29 : rank hope
o29 = 45
```

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Moduli spaces of Riemann surfaces of genus g with n marked points can be dened as smooth Deligneâ€“Mumford stacks (in the algebraic-geometric setting) or as smooth complex orbifolds (in an analytic setting). The latter notion is simpler and will be discussed in the next section. For the time being we dene moduli spaces as sets. Denition 1.6. For 2 âˆ' 2g âˆ' n < 0, the moduli space Mg,n is the set of isomorphism classes of Riemann surfaces of genus g with n marked points. Remark 1.7.Â the moduli space is indeed isomorphic to the automorphism group of the corresponding curve. 13. (A) (B) (C). A double etale cover of X corresponds to (a certain) quadratic extension L/K. I guess every quadratic extension of K is given by taking the root of some element D in K, or by adjoining $(1+\sqrt{D})/2$ to K for some D. I didn't get much further than this though. algebraic-geometry algebraic-curves complex-geometry covering-spaces fundamental-groups.Â This part works for any smooth projective curve X in characteristic $\ne 2$. A double cover $Y\to X$ is given, as you said, by a quadratic extension L of $K={\mathbf C}(X)$. It is an elementary result that such an extension is always given by adjoining a square root z of some $f\in K$: $L=K[z]$ with $z^2=. f$.Â 2g}-1$ double Ã©tale covers, where g is the genus of X.